NPS-EC-02-001

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Pulse Propagation and Bistatic Scattering

by

Lawrence J. Ziomek

26 October 2001

Approved for public release; distribution is unlimited.

Prepared for: Naval Sea Systems Command

REPORT DOCUMENTATION PAGE			*Form Approved OMB No. 0704-0188*
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instruction, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington DC 20503.			
1. AGENCY USE ONLY ***(Leave blank)***	**2. REPORT DATE** 26 October 2001	**3. REPORT TYPE AND DATES COVERED** Technical Report	
4. TITLE AND SUBTITLE Pulse Propagation and Bistatic Scattering			**5. FUNDING NUMBERS** N0002401WR13106
6. AUTHOR(S) Lawrence J. Ziomek			
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Department of Electrical and Computer Engineering, Naval Postgraduate School, Monterey, CA 93943-5121			**8. PERFORMING ORGANIZATION REPORT NUMBER** NPS-EC-02-001
9. SPONSORING /MONITORING AGENCY NAME(S) AND ADDRESS(ES) Naval Sea Systems Command, Program Executive Office for Mine and Undersea Warfare, Attn: PMS403D2, 614 Sicard Street SE STOP 7014, Washington Navy Yard DC 20376-7014			**10. SPONSORING/MONITORING AGENCY REPORT NUMBER**
11. SUPPLEMENTARY NOTES The views expressed in this report are those of the author and do not reflect the official policy or position of the Department of Defense or the United States Government.			
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.			**12b. DISTRIBUTION CODE** A
13. ABSTRACT ***(maximum 200 words)*** A set of pulse-propagation coupling equations is derived. They couple the output electrical signal at a point element in a receive array to the transmitted electrical signal at the input to a transmit array via the complex frequency response of a fluid medium (e.g., air or water). The pulse-propagation coupling equations are based on linear, time-variant, space-variant, filter theory, the principles of complex aperture theory and array theory, and solving a linear wave equation, which includes satisfying all boundary conditions, including the boundary condition at the source. They can be used to accurately model the propagation of small-amplitude acoustic pulses in the ocean for a bistatic scattering problem. Three different bistatic scattering problems are considered: 1) no motion, 2) only the discrete point scatterer is in motion, and 3) all three platforms (the transmitter, discrete point scatterer, and receiver) are in motion. Specific examples on the use of the pulse-propagation coupling equations are given for the three different bistatic scattering problems.			
14. SUBJECT TERMS linear, time-variant, space-variant, filter theory; complex frequency response of the ocean, complex aperture theory, array theory, pulse propagation, pulse-propagation coupling equations, bistatic scattering, scattering amplitude function, time-compression/time-stretch factor			**15. NUMBER OF PAGES** 106
			16. PRICE CODE
17. SECURITY CLASSIFICATION OF REPORT Unclassified	**18. SECURITY CLASSIFICATION OF THIS PAGE** Unclassified	**19. SECURITY CLASSIFICATION OF ABSTRACT** Unclassified	**20. LIMITATION OF ABSTRACT** UL

NSN 7540-01-280-5500

STANDARD FORM 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18 298-102

NAVAL POSTGRADUATE SCHOOL
Monterey, California

RADM. DAVID R. ELLISON
Superintendent

RICHARD S. ELSTER
Provost

This report was sponsored by the Naval Sea Systems Command.
Approved for public release; distribution is unlimited.

This report was prepared by:

Lawrence J. Ziomek

LAWRENCE J. ZIOMEK
Professor
Department of Electrical and
Computer Engineering

Reviewed by:

JEFFREY B. KNORR
Chairman
Department of Electrical and
Computer Engineering

Released by:

DAVID W. NETZER
Associate Provost and
Dean of Research

Table of Contents

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA

26 October 2001

Department of Electrical and
Computer Engineering

Pulse Propagation and Bistatic Scattering

by

Dr. Lawrence J. Ziomek
Naval Postgraduate School
ECE Department
Code EC/ZM
Monterey, CA 93943

Pulse Propagation and Bistatic Scattering

Abstract

A set of pulse-propagation coupling equations is derived. They couple the output electrical signal at a point element in a receive array to the transmitted electrical signal at the input to a transmit array via the complex frequency response of a fluid medium (e.g., air or water). The pulse-propagation coupling equations are based on linear, time-variant, space-variant, filter theory, the principles of complex aperture theory and array theory, and solving a linear wave equation, which includes satisfying all boundary conditions, including the boundary condition at the source. They can be used to accurately model the propagation of small-amplitude acoustic pulses in the ocean for a bistatic scattering problem. Three different bistatic scattering problems are considered: 1) no motion, 2) only the discrete point scatterer is in motion, and 3) all three platforms (the transmitter, discrete point scatterer, and receiver) are in motion. Specific examples on the use of the pulse-propagation coupling equations are given for the three different bistatic scattering problems.

1 Introduction

The main purpose of this report is to establish a mathematical framework that can be used to accurately model the propagation of small-amplitude acoustic pulses in the ocean for a bistatic scattering problem. Since we are dealing with *small-amplitude* acoustic pulses, a *linear* wave equation accurately describes the propagation of sound between source, discrete point scatterer, and receiver. As a result, we treat the propagation of small-amplitude acoustic pulses in the ocean as transmission through a *linear, time-variant, space-variant filter.* Treating the ocean medium as a *linear* filter is valid because we are trying to solve a *linear* wave equation.

Part 2 of this report provides necessary background information on some of the principles of linear, time-variant, space-variant filter theory and how it relates to solving a linear wave equation. In particular, Section 2.1 introduces the *time-variant, space-variant, complex frequency response* of the ocean, which is shown to be the time-harmonic solution of a linear wave equation when the source distribution is a time-harmonic point source. Three different examples are worked out in Section 2.1.

Section 2.2 deals with the main problem of pulse propagation from a linear systems theory point of view. In this section we derive a set of equations that we refer to as the *pulse-propagation coupling equations.* They couple the output electrical signal at a point element in a receive array to the transmitted electrical signal at the input to a transmit array via the complex frequency response of a fluid medium (e.g., air or water). The pulse-propagation coupling equations are based on *linear, time-variant, space-variant, filter theory,* the principles of *complex aperture theory* and *array theory,* and solving a *linear wave equation,* which includes satisfying all boundary conditions, including the boundary condition at the source. They provide a consistent, logical, and straightforward mathematical framework for the solution of small-amplitude, acoustic, pulse-propagation problems. The main features of the pulse-propagation coupling equations are as follows: 1) transmitted electrical signals are modeled as *amplitude-and-angle-modulated carriers,* 2) both the transmit and receive apertures are modeled as *volume, conformal arrays* composed of *unevenly-spaced, complex-weighted, point elements* (this type of model for both of the apertures allows for maximum flexibility), 3) the complex weights are frequency dependent and allow for beamforming, 4) the performance of the point elements in both the transmit and receive arrays are characterized by frequency-dependent, transmitter and receiver sensitivity functions, and 5) the solution of a linear wave equation is given by the complex frequency response of the fluid medium. It is important to note that attention to all proper units of measurement are taken into account in

order to ensure the accurate prediction of signal strength levels at each element in a receive array. This is especially important, for example, in order to obtain accurate probability of detection results.

Part 3 of this report is devoted to the derivation of the complex frequency response of the ocean for three different bistatic scattering problems. Scatter from a discrete point scatterer is modeled via the *scattering amplitude function*, which is a complex function (magnitude and phase) and is, in general, a function of frequency, the direction of wave propagation from the source to the scatterer, and the direction of wave propagation from the scatterer to the receiver. In addition to the scattering amplitude function, frequency-dependent attenuation is taken into account in order to model the propagation of sound from transmitter to discrete point scatterer, and from discrete point scatterer to receiver. Section 3.1 is devoted to the first bistatic scattering problem, which involves *no* motion - the transmitter, the discrete point scatterer, and the receiver are *not* in motion. Section 3.2 is devoted to the second bistatic scattering problem where only the discrete point scatterer is in motion, and Section 3.3 is devoted to the third bistatic scattering problem where all three platforms are in motion. The dimensionless, *time-compression / time-stretch factor* is discussed in Sections 3.2 and 3.3 due to the consideration of motion. The time-compression / time-stretch factor takes into account the relativistic effects of motion and provides for accurate time delay and Doppler shift values. In addition to other examples, specific examples on the use of the pulse-propagation coupling equations are given in all three sections of Part 3 for the three different bistatic scattering problems.

2 The Ocean Medium As A Linear, Time-Variant, Space-Variant Filter

2.1 Complex Frequency Response Of The Ocean

The propagation of *small-amplitude* acoustic signals in the ocean medium can be described by the following *linear*, three-dimensional, inhomogeneous wave equation:

$$\nabla^2 y_M(t,\mathbf{r}) - \frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2} y_M(t,\mathbf{r}) = x_M(t,\mathbf{r}), \tag{2.1-1}$$

where $y_M(t,\mathbf{r})$ is the *velocity potential* at time t and position $\mathbf{r} = (x,y,z)$ with units of squared-meters per second, $x_M(t,\mathbf{r})$ is the *source distribution* at time t and position $\mathbf{r}$ with units of inverse seconds, and $c(\mathbf{r})$ is the speed of sound in meters per second, shown as a function of position $\mathbf{r}$. The source distribution represents the volume flow rate (source strength) per unit volume of fluid. The solution of (2.1-1) can be obtained by treating the ocean medium as a *linear, time-variant, space-variant filter*. Treating the ocean medium as a *linear* filter is valid because we are trying to solve a *linear* wave equation. The time-variant property of the filter allows for motion and the possibility that the properties of the ocean medium may change with time. The space-variant property of the filter allows for the presence of boundaries, both the speed of sound and the ambient (equilibrium) density of the ocean medium as functions of position (spatial coordinates), and discrete point scatterers.

Figure 2.1-1 is an illustration of a linear, time-variant, space-variant, ocean-medium filter with input-output relationship given by [1]

$$\boxed{y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_M(t_0,\mathbf{r}_0) h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0)\,dt_0\,d\mathbf{r}_0,} \tag{2.1-2}$$

where the source distribution $x_M(t,\mathbf{r})$ is the input acoustic signal to the filter, $h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0)$ is the *time-variant, space-variant, impulse response* (*Green's function*) of the filter, that is, it is the

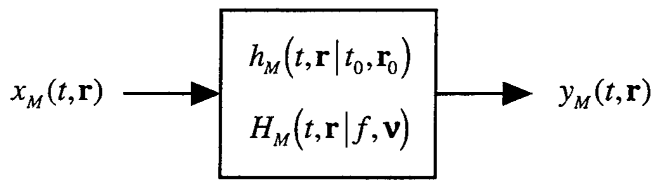

Figure 2.1-1. Illustration of a linear, time-variant, space-variant, ocean-medium filter.

response of the filter at time t and position $\mathbf{r} = (x, y, z)$ due to the application of a unit-amplitude impulse at time t_0 and position $\mathbf{r}_0 = (x_0, y_0, z_0)$, the velocity potential $y_M(t, \mathbf{r})$ is the output acoustic signal from the filter, and $d\mathbf{r}_0 = dx_0\, dy_0\, dz_0$. Therefore, the right-hand side of (2.1-2) is shorthand notation for a four-fold integral. Note that (2.1-2) is *not* a multidimensional convolution integral. Equation (2.1-2) is the solution of the linear wave equation given by (2.1-1).

Also shown in Fig. 2.1-1 is the *time-variant, space-variant, transfer function* $H_M(t, \mathbf{r} \,|\, f, \mathbf{v})$ of the ocean-medium filter defined by [1]

$$\boxed{\begin{aligned} H_M(t,\mathbf{r}\,|\,f,\mathbf{v}) &\triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0)\exp[-j2\pi f(t-t_0)]\exp[+j2\pi\mathbf{v}\bullet(\mathbf{r}-\mathbf{r}_0)]\,dt_0\,d\mathbf{r}_0 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0)\exp[-j2\pi f(t-t_0)]\,dt_0\exp[+j2\pi\mathbf{v}\bullet(\mathbf{r}-\mathbf{r}_0)]\,d\mathbf{r}_0 \\ &= \int_{-\infty}^{\infty} H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0)\exp[+j2\pi\mathbf{v}\bullet(\mathbf{r}-\mathbf{r}_0)]\,d\mathbf{r}_0, \end{aligned}}$$

(2.1-3)

where f represents input (transmitted) frequency components in hertz, $\mathbf{v} = (f_X, f_Y, f_Z)$ is a three-dimensional vector whose components are input (transmitted) *spatial frequencies* in cycles per meter in the X, Y, and Z directions, respectively, $d\mathbf{r}_0 = dx_0\, dy_0\, dz_0$, and

$$\boxed{H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0) \triangleq \int_{-\infty}^{\infty} h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0)\exp[-j2\pi f(t-t_0)]\,dt_0} \tag{2.1-4}$$

is the *time-variant, space-variant, complex frequency response* of the ocean at frequency f hertz. Similarly [1],

$$\boxed{\begin{aligned} h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0) &\triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_M(t,\mathbf{r}\,|\,f,\mathbf{v})\exp[+j2\pi f(t-t_0)]\exp[-j2\pi\mathbf{v}\bullet(\mathbf{r}-\mathbf{r}_0)]\,df\,d\mathbf{v} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_M(t,\mathbf{r}\,|\,f,\mathbf{v})\exp[-j2\pi\mathbf{v}\bullet(\mathbf{r}-\mathbf{r}_0)]\,d\mathbf{v}\exp[+j2\pi f(t-t_0)]\,df \\ &= \int_{-\infty}^{\infty} H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0)\exp[+j2\pi f(t-t_0)]\,df, \end{aligned}}$$

(2.1-5)

where $d\mathbf{v} = df_X\, df_Y\, df_Z$. From the right-hand side of (2.1-5), it can be seen that the complex frequency response $H_M(t, \mathbf{r} \,|\, f, \mathbf{r}_0)$ can be obtained from the transfer function $H_M(t, \mathbf{r} \,|\, f, \mathbf{v})$ as follows:

$$\boxed{H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0) = \int_{-\infty}^{\infty} H_M(t,\mathbf{r}\,|\,f,\mathbf{v})\exp[-j2\pi\mathbf{v}\bullet(\mathbf{r}-\mathbf{r}_0)]\,d\mathbf{v}.} \tag{2.1-6}$$

Equation (2.1-6) is the inverse of the last relationship shown in (2.1-3).

The strategy that we are going to follow is to first find time-harmonic solutions of the linear wave equation for various bistatic scattering problems, and then, by using Fourier transform techniques, we will obtain the corresponding pulse solutions. Recall that a time-harmonic acoustic field is one whose value at any time and point in space depends on a single frequency component.

For convenience, let us rewrite (2.1-2) as follows:

$$y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_M(t_0,\boldsymbol{\zeta}) h_M(t,\mathbf{r}\,|\,t_0,\boldsymbol{\zeta})\,dt_0\,d\boldsymbol{\zeta}. \tag{2.1-7}$$

Let the source distribution $x_M(t,\mathbf{r})$ be *motionless* and *time-harmonic*, that is,

$$x_M(t,\mathbf{r}) = x_{f,M}(\mathbf{r})\exp(+j2\pi ft), \tag{2.1-8}$$

where $x_{f,M}(\mathbf{r})$ is the spatial-dependent part (with units of inverse seconds) of the motionless, time-harmonic source distribution. Substituting (2.1-8) into (2.1-7) yields

$$y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty} x_{f,M}(\boldsymbol{\zeta}) \int_{-\infty}^{\infty} h_M(t,\mathbf{r}\,|\,t_0,\boldsymbol{\zeta})\exp(+j2\pi ft_0)\,dt_0\,d\boldsymbol{\zeta}, \tag{2.1-9}$$

and upon multiplying the right-hand side of (2.1-9) by

$$\exp(-j2\pi ft)\exp(+j2\pi ft) = 1,$$

we obtain the following expression for the output acoustic signal (velocity potential) from a linear, *time-variant*, *space-variant*, ocean-medium filter:

$$\boxed{y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty} x_{f,M}(\boldsymbol{\zeta}) H_M(t,\mathbf{r}\,|\,f,\boldsymbol{\zeta})\,d\boldsymbol{\zeta}\exp(+j2\pi ft),} \tag{2.1-10}$$

where $H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0)$ is the *time-variant, space-variant, complex frequency response* of the ocean at frequency f hertz given by (2.1-4).

If the source distribution $x_M(t,\mathbf{r})$ is a *motionless, time-harmonic, point source* with units of inverse seconds, then (2.1-8) is still applicable but

$$x_{f,M}(\mathbf{r}) = S_0\delta(\mathbf{r}-\mathbf{r}_0), \tag{2.1-11}$$

where S_0 is the *source strength* in cubic meters per second and the impulse function $\delta(\mathbf{r}-\mathbf{r}_0)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_0 = (x_0, y_0, z_0)$. Substituting (2.1-11) into (2.1-10) yields

$$\boxed{y_M(t,\mathbf{r}) = S_0 H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0)\exp(+j2\pi ft),} \tag{2.1-12}$$

where $H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0)$ is given by (2.1-4).

Different forms of the input-output relationship given by (2.1-2) and the time-variant, space-variant, complex frequency response given by (2.1-4) can be obtained by considering the following three examples.

Example 2.1-1 Time-Invariant, Space-Invariant, Ocean-Medium Filters

If the ocean-medium filter is *time-invariant* and *space-invariant*, then [1]

$$\boxed{h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0) = h_M(t-t_0,\mathbf{r}-\mathbf{r}_0),} \tag{2.1-13}$$

that is, the impulse response depends only on the *time difference* $\tau = t - t_0$ and the *vector spatial difference* $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$. The time difference τ is a measure of how long ago the impulse was applied, or the travel time between source and field points. The vector spatial difference $\mathbf{R}$ corresponds to the distance and direction between source and field points. The magnitude of the vector spatial difference $|\mathbf{R}| = |\mathbf{r} - \mathbf{r}_0|$ is a measure of how far away the impulse was applied, or the distance traveled between source and field points. Substituting (2.1-13) into (2.1-2) yields

$$\boxed{y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_M(t_0,\mathbf{r}_0)h_M(t-t_0,\mathbf{r}-\mathbf{r}_0)\,dt_0\,d\mathbf{r}_0,} \tag{2.1-14}$$

which is a *multidimensional convolution integral* as expected. In addition, substituting (2.1-13) into (2.1-4) yields

$$H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0) = \int_{-\infty}^{\infty} h_M(t-t_0,\mathbf{r}-\mathbf{r}_0)\exp[-j2\pi f(t-t_0)]\,dt_0. \tag{2.1-15}$$

If we let $\tau = t - t_0$, then $dt_0 = -d\tau$ and (2.1-15) reduces to

$$\boxed{\begin{aligned} H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0) &= \int_{-\infty}^{\infty} h_M(\tau,\mathbf{r}-\mathbf{r}_0)\exp(-j2\pi f\tau)\,d\tau \\ &= F_\tau\{h_M(\tau,\mathbf{r}-\mathbf{r}_0)\} \\ &= H_M(f,\mathbf{r}-\mathbf{r}_0). \end{aligned}} \tag{2.1-16}$$

If the source distribution is *motionless* and *time-harmonic* [see (2.1-8)] and the ocean-medium filter is *time-invariant* and *space-invariant*, then substituting (2.1-16) into (2.1-10) yields the following expression for the output acoustic signal (velocity potential) from the ocean-medium filter:

$$\boxed{y_M(t,\mathbf{r}) = y_{f,M}(\mathbf{r})\exp(+j2\pi ft),} \tag{2.1-17}$$

where

$$\boxed{y_{f,M}(\mathbf{r}) = x_{f,M}(\mathbf{r}) \underset{\mathbf{r}}{*} H_M(f,\mathbf{r}) = \int_{-\infty}^{\infty} x_{f,M}(\boldsymbol{\zeta})H_M(f,\mathbf{r}-\boldsymbol{\zeta})\,d\boldsymbol{\zeta}} \tag{2.1-18}$$

and the *time-invariant, space-invariant, complex frequency response* of the ocean at frequency f hertz is given by [see (2.1-16)]

$$\boxed{H_M(f,\mathbf{r}-\mathbf{r}_0) = F_\tau\{h_M(\tau,\mathbf{r}-\mathbf{r}_0)\} = \int_{-\infty}^{\infty} h_M(\tau,\mathbf{r}-\mathbf{r}_0)\exp(-j2\pi f\tau)\,d\tau.} \tag{2.1-19}$$

Furthermore, if the source distribution is a *motionless, time-harmonic, point source,* then (2.1-8) and (2.1-11) are applicable, and upon substituting (2.1-11) into (2.1-18), we obtain

$$\boxed{y_{f,M}(\mathbf{r}) = S_0 H_M(f,\mathbf{r}-\mathbf{r}_0),} \tag{2.1-20}$$

where $H_M(f,\mathbf{r}-\mathbf{r}_0)$ is given by (2.1-19).

For example, the impulse response of an *ideal* (*nonviscous*), *unbounded, homogeneous, ocean medium* is given by [2, 3]

$$\boxed{h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0) = h_M(t-t_0,\mathbf{r}-\mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}_0|}\delta\left(t-t_0-\frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right).} \tag{2.1-21}$$

Equation (2.1-21) indicates that the ocean-medium filter is *time-invariant* (no motion and the properties of the medium between source and field points do not change with time) and space-invariant (unbounded, homogeneous medium with no discrete point scatterers between source and field points). If we let $\tau = t - t_0$, then substituting (2.1-21) into (2.1-19) yields the following expression for the *time-invariant, space-invariant, complex frequency response* of the ocean at frequency f hertz:

$$\boxed{H_M(f,\mathbf{r}-\mathbf{r}_0) = -\frac{\exp(-jk|\mathbf{r}-\mathbf{r}_0|)}{4\pi|\mathbf{r}-\mathbf{r}_0|},} \tag{2.1-22}$$

where

$$k = 2\pi f/c = 2\pi/\lambda \tag{2.1-23}$$

is the *wavenumber* with units of radians per meter, c is the *constant* speed of sound of the homogeneous ocean medium in meters per second, and λ is the *wavelength* in meters. Note that $H_M(f,\mathbf{r}-\mathbf{r}_0)$ given by (2.1-22) is an *exact* solution of the following lossless, inhomogeneous, Helmholtz equation [4, 5]:

$$\nabla^2 H_M(f,\mathbf{r}-\mathbf{r}_0) + k^2 H_M(f,\mathbf{r}-\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0). \tag{2.1-24}$$

Also note that $h_M(t-t_0,\mathbf{r}-\mathbf{r}_0)$ given by (2.1-21) can be used in (2.1-14) and that $H_M(f,\mathbf{r}-\mathbf{r}_0)$ given by (2.1-22) can be used in both (2.1-18) and (2.1-20), and that

$$H_M(f,\mathbf{r}) = -\frac{\exp(-jk|\mathbf{r}|)}{4\pi|\mathbf{r}|}. \tag{2.1-25}$$

Example 2.1-2 Time-Invariant, Space-Variant, Ocean-Medium Filters

If the ocean-medium filter is *time-invariant* and *space-variant*, then [1]

$$\boxed{h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0) = h_M(t-t_0,\mathbf{r}\,|\,\mathbf{r}_0),} \tag{2.1-26}$$

that is, the impulse response depends on the *time difference* $\tau = t - t_0$. The time difference τ is a measure of how long ago the impulse was applied, or the travel time between source and field points. Substituting (2.1-26) into (2.1-2) yields

$$\boxed{y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_M(t_0,\mathbf{r}_0)h_M(t-t_0,\mathbf{r}\,|\,\mathbf{r}_0)dt_0\,d\mathbf{r}_0\,.} \tag{2.1-27}$$

In addition, substituting (2.1-26) into (2.1-4) yields

$$H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0) = \int_{-\infty}^{\infty} h_M(t-t_0,\mathbf{r}\,|\,\mathbf{r}_0)\exp[-j2\pi f(t-t_0)]dt_0. \tag{2.1-28}$$

If we let $\tau = t - t_0$, then $dt_0 = -d\tau$ and (2.1-28) reduces to

$$\boxed{\begin{aligned} H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0) &= \int_{-\infty}^{\infty} h_M(\tau,\mathbf{r}\,|\,\mathbf{r}_0)\exp(-j2\pi f\tau)\,d\tau \\ &= F_\tau\{h_M(\tau,\mathbf{r}\,|\,\mathbf{r}_0)\} \\ &= H_M(f,\mathbf{r}\,|\,\mathbf{r}_0). \end{aligned}} \tag{2.1-29}$$

If the source distribution is *motionless* and *time-harmonic* [see (2.1-8)] and the ocean-medium filter is *time-invariant* and *space-variant*, then substituting (2.1-29) into (2.1-10) yields the following expression for the output acoustic signal (velocity potential) from the ocean-medium filter:

$$\boxed{y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty} x_{f,M}(\boldsymbol{\zeta})H_M(f,\mathbf{r}\,|\,\boldsymbol{\zeta})d\boldsymbol{\zeta}\exp(+j2\pi ft),} \tag{2.1-30}$$

where the *time-invariant, space-variant, complex frequency response* of the ocean at frequency f hertz is given by [see(2.1-29)]

$$\boxed{H_M(f,\mathbf{r}\,|\,\mathbf{r}_0) = F_\tau\{h_M(\tau,\mathbf{r}\,|\,\mathbf{r}_0)\} = \int_{-\infty}^{\infty} h_M(\tau,\mathbf{r}\,|\,\mathbf{r}_0)\exp(-j2\pi f\tau)\,d\tau.} \tag{2.1-31}$$

Furthermore, if the source distribution is a *motionless, time-harmonic, point source,* then (2.1-8) and (2.1-11) are applicable, and upon substituting either (2.1-11) into (2.1-30), or (2.1-29) into (2.1-12), we obtain

$$\boxed{y_M(t,\mathbf{r}) = S_0 H_M(f,\mathbf{r}\,|\,\mathbf{r}_0)\exp(+j2\pi ft),} \tag{2.1-32}$$

where $H_M(f,\mathbf{r}\,|\,\mathbf{r}_0)$ is given by (2.1-31). Several different examples of $H_M(f,\mathbf{r}\,|\,\mathbf{r}_0)$ are given next.

A simple waveguide model of the ocean is illustrated in Fig. 2.1-2 where $\rho_1 c_1$, $\rho_2 c_2$, and $\rho_3 c_3$ are the characteristic impedances of fluid media I (air), II (seawater), and III (ocean bottom), respectively. The parameters ρ_i and c_i, $i = 1,2,3$, are the constant ambient (equilibrium) densities (in kilograms per cubic meter) and speeds of sound (in meters per second) in each of the three fluid media, respectively. Both the ocean surface and ocean bottom are modeled as plane, parallel boundaries, and the ocean is D meters deep. The source distribution is a *motionless, time-harmonic, point source* [see (2.1-8) and (2.1-11)] located in medium II at horizontal range $r = 0$ meters and depth $y = y_0$ meters where the cylindrical coordinates (r,ϕ,y) are illustrated in Fig. 2.1-3.

If the ocean surface is modeled as an *ideal pressure-release boundary* and the ocean bottom is modeled as an *ideal rigid boundary,* then it can be shown that the *time-invariant, space-variant, complex frequency response* of the ocean at frequency f hertz is given by [6]

$$\boxed{H_M(f,\mathbf{r}\,|\,\mathbf{r}_0) = H_M(f,r,\phi,y\,|\,y_0) = -j\frac{1}{2D}\sum_{n=0}^{N_p-1} \sin\left(k_{Y_{2,n}} y_0\right) H_0^{(2)}\left(k_{r_{2,n}} r\right)\sin\left(k_{Y_{2,n}} y\right), \qquad 0 \le y \le D,} \tag{2.1-33}$$

where

$$N_p = \mathrm{INT}\left(\frac{1}{2}\left[\frac{4Df}{c_2} - 1\right]\right) + 1 \tag{2.1-34}$$

is the total number of propagating normal modes excited by the source frequency f hertz, $H_0^{(2)}(\bullet)$ is the zeroth-order Hankel function of the second kind,

$$k_{r_{2,n}} = \begin{cases} k_2\sqrt{1-(f_n/f)^2}, & f \ge f_n, \\ -jk_2\sqrt{(f_n/f)^2 - 1}, & f < f_n, \end{cases} \tag{2.1-35}$$

is the propagation-vector component in the horizontal radial direction, in radians per meter, of the nth normal mode,

$$k_2 = 2\pi f/c_2 = 2\pi/\lambda_2 \tag{2.1-36}$$

is the wavenumber in medium II in radians per meter,

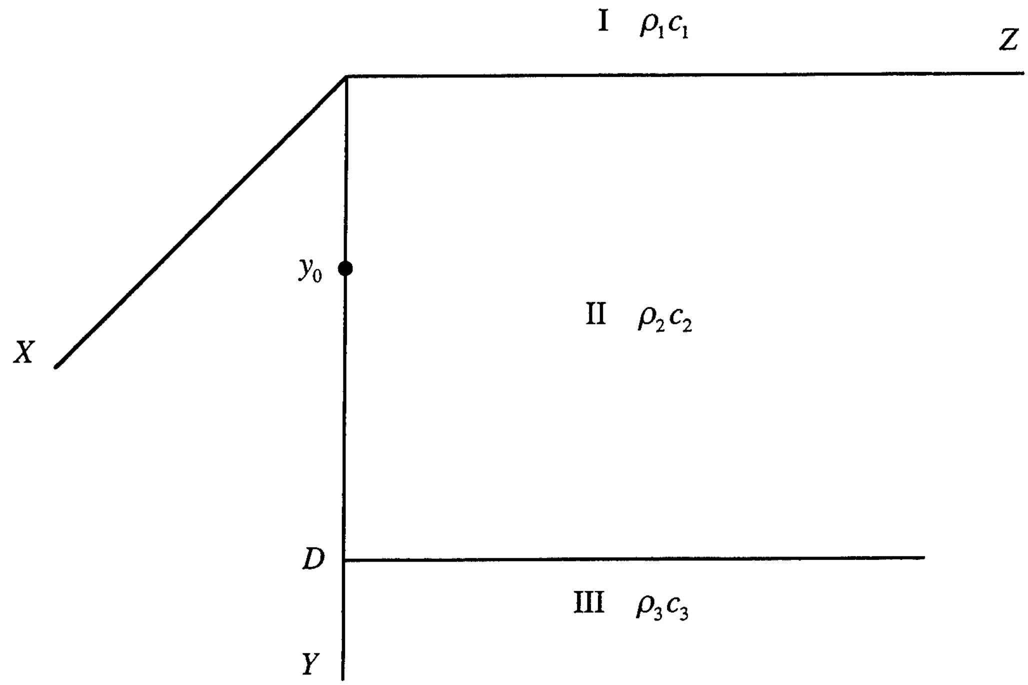

Figure 2.1-2. Simple waveguide model of the ocean where $\rho_1 c_1$, $\rho_2 c_2$, and $\rho_3 c_3$ are the characteristic impedances of fluid media I (air), II (seawater), and III (ocean bottom), respectively. The source distribution is a motionless, time-harmonic, point source located in medium II at horizontal range $r = 0$ meters and depth $y = y_0$ meters.

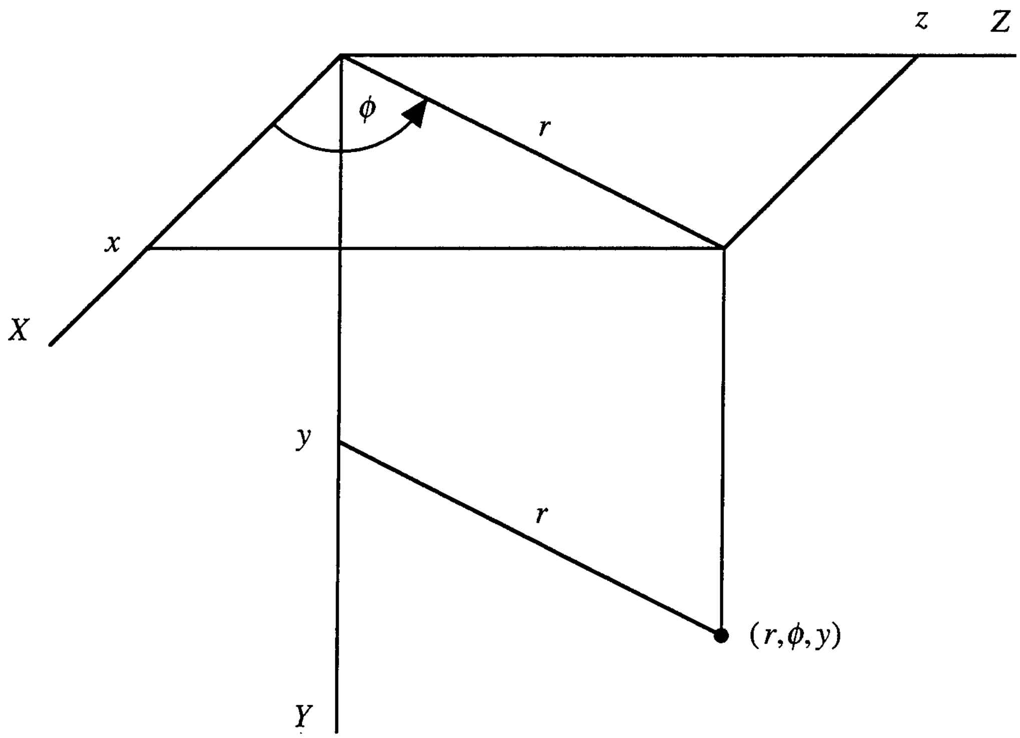

Figure 2.1-3. The cylindrical coordinates (r,ϕ,y).

$$f_n = \frac{(2n+1)c_2}{4D} \tag{2.1-37}$$

is the cutoff frequency in hertz of the nth normal mode, and

$$k_{Y_{2,n}} = \frac{(2n+1)\pi}{2D} \tag{2.1-38}$$

is the propagation-vector component in the depth direction, in radians per meter, of the nth normal mode. Note that in the far-field, that is, when $k_{r_{2,n}} r >> 1$,

$$H_0^{(2)}\left(k_{r_{2,n}} r\right) \approx \sqrt{\frac{2}{\pi k_{r_{2,n}} r}} \exp\left[-j\left(k_{r_{2,n}} r - \frac{\pi}{4}\right)\right], \qquad k_{r_{2,n}} r >> 1. \tag{2.1-39}$$

By inspecting (2.1-33), it can be seen that the complex frequency response depends on the actual values of the depth coordinates of the field point (y) and the point source (y_0) and *not* on their difference. This is an indication that the medium is *space-variant* in the depth direction.

A second example of a time-invariant, space-variant, complex frequency response can be obtained by reconsidering the simple waveguide model of the ocean illustrated in Fig. 2.1-2. If the ocean surface is once again modeled as an *ideal pressure-release boundary* as before, and if the ocean bottom is now modeled as *the boundary between two different fluid media* instead of being modeled as an ideal rigid boundary, then it can be shown that the *time-invariant, space-variant, complex frequency response* of the ocean at frequency f hertz for a *fast fluid-bottom* is given by [7]

$$\boxed{H_M(f,\mathbf{r}|\mathbf{r}_0) = H_M(f,r,\phi,y|y_0) = -j\frac{1}{4}\sum_{n=0}^{N_t-1} E_n^{-1}\sin\left(k_{Y_{2,n}} y_0\right) H_0^{(2)}\left(k_{r_{2,n}} r\right)\sin\left(k_{Y_{2,n}} y\right), \qquad 0 \le y \le D,} \tag{2.1-40}$$

where

$$N_t = \mathrm{INT}\left(\frac{1}{2}\left[\frac{4Df\cos\beta_c}{c_2} - 1\right]\right) + 1 \tag{2.1-41}$$

is the total number of trapped (propagating) normal modes excited by the source frequency f hertz,

$$\beta_c = \sin^{-1}(c_2/c_3), \qquad c_3 > c_2, \tag{2.1-42}$$

is the critical angle of incidence,

$$E_n = \frac{2k_{Y_{2,n}} D - \sin\left(2k_{Y_{2,n}} D\right) - 2(\rho_2/\rho_3)^3 \tan\left(k_{Y_{2,n}} D\right)\sin^2\left(k_{Y_{2,n}} D\right)}{4k_{Y_{2,n}}} \tag{2.1-43}$$

is the "energy" in meters of the nth eigenfunction, $H_0^{(2)}(\bullet)$ is the zeroth-order Hankel function of the second kind,

$$k_{r_{2,n}} = k_2 \sin \beta_{2,n} \tag{2.1-44}$$

is the propagation-vector component in the horizontal radial direction, in radians per meter, of the nth normal mode (see Fig. 2.1-4),

$$k_{Y_{2,n}} = k_2 \cos \beta_{2,n} \tag{2.1-45}$$

is the propagation-vector component in the depth direction, in radians per meter, of the nth normal mode (see Fig. 2.1-4),

$$k_2 = 2\pi f / c_2 = 2\pi / \lambda_2 \tag{2.1-46}$$

is the wavenumber in medium II in radians per meter, and the angles of propagation $\beta_{2,n}$, $n = 0,1,\ldots,N_t - 1$, (see Fig. 2.1-4) are the roots of the following transcendental equation:

$$\tan\left(\frac{2\pi f}{c_2} D \cos \beta_{2,n}\right) + \frac{\rho_3 c_3 \cos \beta_{2,n}}{\rho_2 c_2 \sqrt{\left(\sin \beta_{2,n} / \sin \beta_c\right)^2 - 1}} = 0, \qquad c_3 > c_2, \quad \beta_c < \beta_{2,n} < \pi/2. \tag{2.1-47}$$

The far-field approximation of $H_0^{(2)}(\bullet)$ is given by (2.1-39). By inspecting (2.1-40), it can be seen that the complex frequency response depends on the actual values of the depth coordinates of the field point (y) and the point source (y_0) and *not* on their difference. This is an indication that the medium is *space-variant* in the depth direction.

A third and final example of a time-invariant, space-variant, complex frequency response can be obtained by reconsidering the simple waveguide model of the ocean illustrated in Fig. 2.1-2. If both the ocean surface and the ocean bottom are modeled as *boundaries between two different fluid media*, then it can be shown that the *time-invariant, space-variant, complex frequency response* of the ocean at frequency f hertz is given by [8]

$$\boxed{\begin{aligned} H_M(f,\mathbf{r}|\mathbf{r}_0) &= H_M(f,r,\phi,y|y_0) \\ &= +j\frac{1}{4\pi}\int_0^\infty \frac{F(k_r,y_0)}{k_{Y_2}}\left\{R_{21}\exp\left[-jk_{Y_2}(y+y_0)\right] + \exp\left[+jk_{Y_2}(y-y_0)\right]\right\} \\ &\quad \times J_0(k_r r) k_r \, dk_r, \qquad 0 \le y \le y_0, \end{aligned}} \tag{2.1-48}$$

when the field point is at a depth between the ocean surface and the point source ($0 \le y \le y_0$), and

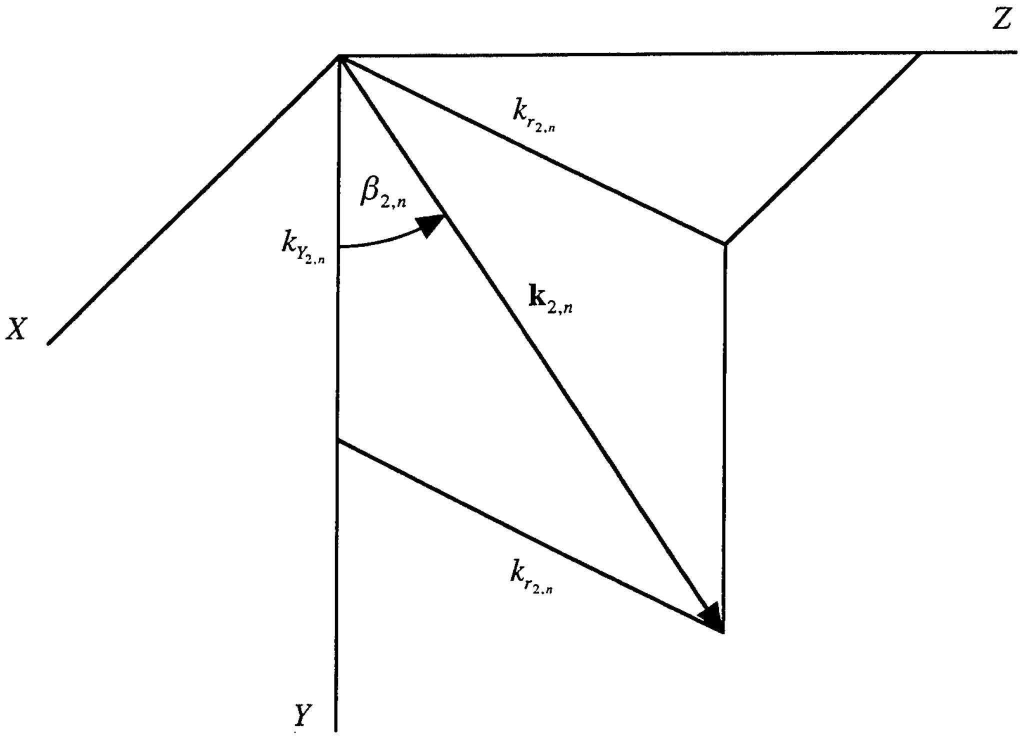

Figure 2.1-4. The propagation vector $\mathbf{k}_{2,n}$ of the nth trapped (propagating) normal mode in medium II. Note that $|\mathbf{k}_{2,n}| = k_2 = 2\pi f/c_2$. Also shown are the angle of propagation $\beta_{2,n}$ and the scalar propagation-vector components $k_{r_{2,n}}$ and $k_{Y_{2,n}}$ in the horizontal range r and depth Y directions, respectively.

$$\boxed{\begin{aligned}H_M(f,\mathbf{r}|\mathbf{r}_0) &= H_M(f,r,\phi,y|y_0)\\ &= +j\frac{1}{4\pi}\int_0^\infty \frac{F(k_r,y)}{k_{Y_2}}\left\{R_{21}\exp\left[-jk_{Y_2}(y+y_0)\right]+\exp\left[-jk_{Y_2}(y-y_0)\right]\right\}\\ &\qquad\times J_0(k_r r)k_r\,dk_r, \qquad y_0 \le y \le D,\end{aligned}}$$

(2.1-49)

when the field point is at a depth between the point source and the ocean bottom ($y_0 \le y \le D$), where

$$F(k_r,y) = \frac{1+R_{23}\exp\left[-j2k_{Y_2}(D-y)\right]}{1-R_{21}R_{23}\exp\left(-j2k_{Y_2}D\right)}, \tag{2.1-50}$$

$$F(k_r,y_0) = F(k_r,y)\big|_{y=y_0}, \tag{2.1-51}$$

$$R_{21} = \frac{\rho_1 k_{Y_2} - \rho_2 k_{Y_1}}{\rho_1 k_{Y_2} + \rho_2 k_{Y_1}} \tag{2.1-52}$$

and

$$R_{23} = \frac{\rho_3 k_{Y_2} - \rho_2 k_{Y_3}}{\rho_3 k_{Y_2} + \rho_2 k_{Y_3}} \tag{2.1-53}$$

are the plane-wave velocity potential reflection coefficients at the ocean surface and ocean bottom, respectively,

$$k_{Y_i} = \begin{cases} \sqrt{k_i^2 - k_r^2}, & k_r^2 \le k_i^2, \quad i = 1,2,3, \\ -j\sqrt{k_r^2 - k_i^2}, & k_r^2 > k_i^2, \quad i = 1,2,3, \end{cases} \tag{2.1-54}$$

are the propagation-vector components in the depth direction, in radians per meter, in media I, II, and III, respectively,

$$k_i = 2\pi f/c_i, \qquad i = 1,2,3, \tag{2.1-55}$$

are the wavenumbers in radians per meter in media I, II, and III, respectively, and $J_0(\bullet)$ is the zeroth-order Bessel function of the first kind. Note that in the far-field, that is, when $k_r r >> 1$,

$$J_0(k_r r) \approx \sqrt{\frac{2}{\pi k_r r}}\cos\left(k_r r - \frac{\pi}{4}\right), \qquad k_r r >> 1. \tag{2.1-56}$$

Note that the function $F(k_r, y)$ in the integrand of (2.1-49) is *not* evaluated at $y = y_0$. By inspecting (2.1-48) and (2.1-49), it can be seen that the complex frequency response depends on the actual values of the depth coordinates of the field point (y) and the point source (y_0) and *not* on their difference. This is an indication that the medium is *space-variant* in the depth direction.

Example 2.1-3 Time-Variant, Space-Invariant, Ocean-Medium Filters

If the ocean-medium filter is *time-variant* and *space-invariant*, then [1]

$$\boxed{h_M(t,\mathbf{r} \,|\, t_0,\mathbf{r}_0) = h_M(t,\mathbf{r}-\mathbf{r}_0 \,|\, t_0),} \tag{2.1-57}$$

that is, the impulse response depends on the *vector spatial difference* $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$. The vector spatial difference $\mathbf{R}$ corresponds to the distance and direction between source and field points. The magnitude of the vector spatial difference $|\mathbf{R}| = |\mathbf{r} - \mathbf{r}_0|$ is a measure of how far away the impulse was applied, or the distance traveled between source and field points. Substituting (2.1-57) into (2.1-2) yields

$$\boxed{y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_M(t_0,\mathbf{r}_0) h_M(t,\mathbf{r}-\mathbf{r}_0 \,|\, t_0)\, dt_0\, d\mathbf{r}_0.} \tag{2.1-58}$$

In addition, substituting (2.1-57) into (2.1-4) yields

$$\boxed{\begin{aligned} H_M(t,\mathbf{r} \,|\, f,\mathbf{r}_0) &= \int_{-\infty}^{\infty} h_M(t,\mathbf{r}-\mathbf{r}_0 \,|\, t_0) \exp[-j2\pi f(t-t_0)]\, dt_0 \\ &= H_M(t,\mathbf{r}-\mathbf{r}_0 \,|\, f). \end{aligned}} \tag{2.1-59}$$

If the source distribution is *motionless* and *time-harmonic* [see (2.1-8)] and the ocean-medium filter is *time-variant* and *space-invariant*, then substituting (2.1-59) into (2.1-10) yields the following expression for the output acoustic signal (velocity potential) from the ocean-medium filter:

$$\boxed{\begin{aligned} y_M(t,\mathbf{r}) &= \int_{-\infty}^{\infty} x_{f,M}(\boldsymbol{\zeta}) H_M(t,\mathbf{r}-\boldsymbol{\zeta} \,|\, f)\, d\boldsymbol{\zeta} \exp(+j2\pi ft) \\ &= \left[x_{f,M}(\mathbf{r}) \underset{\mathbf{r}}{*} H_M(t,\mathbf{r} \,|\, f)\right] \exp(+j2\pi ft), \end{aligned}} \tag{2.1-60}$$

where the *time-variant, space-invariant, complex frequency response* of the ocean at frequency f hertz is given by [see (2.1-59)]

$$\boxed{H_M(t,\mathbf{r}-\mathbf{r}_0 \,|\, f) = \int_{-\infty}^{\infty} h_M(t,\mathbf{r}-\mathbf{r}_0 \,|\, t_0) \exp[-j2\pi f(t-t_0)]\, dt_0.} \tag{2.1-61}$$

Furthermore, if the source distribution is a *motionless, time-harmonic, point source,* then (2.1-8) and (2.1-11) are applicable, and upon substituting either (2.1-11) into (2.1-60), or (2.1-59) into (2.1-12), we obtain

$$\boxed{y_M(t,\mathbf{r}) = S_0 H_M\left(t,\mathbf{r}-\mathbf{r}_0 \,|\, f\right)\exp(+j2\pi ft),} \tag{2.1-62}$$

where $H_M\left(t,\mathbf{r}-\mathbf{r}_0 \,|\, f\right)$ is given by (2.1-61).

2.2 Pulse Propagation In The Ocean

Before we can discuss the pulse-propagation problem in this section, we need to introduce two *coupling equations*. The first coupling equation is the *transmit-aperture coupling equation* and is given by (see [9] and Fig. 2.2-1)

$$\boxed{x_M(t,\mathbf{r}) = \int_{-\infty}^{\infty} X(f,\mathbf{r}) A_T(f,\mathbf{r}) \exp(+j2\pi ft)\,df,} \tag{2.2-1}$$

where $x_M(t,\mathbf{r})$ is the input acoustic signal (source distribution) to the fluid medium with units of inverse seconds (volume flow rate per unit volume of fluid) at time t and position $\mathbf{r}=(x,y,z)$ of the transmit aperture,

$$X(f,\mathbf{r}) = F_t\{x(t,\mathbf{r})\} = \int_{-\infty}^{\infty} x(t,\mathbf{r})\exp(-j2\pi ft)\,dt \tag{2.2-2}$$

is the complex frequency spectrum of the input electrical signal at position $\mathbf{r}=(x,y,z)$ of the transmit aperture with units of volts per hertz, $x(t,\mathbf{r})$ is the input electrical signal to the transmit aperture at time t and position $\mathbf{r}=(x,y,z)$ with units of volts, and $A_T(f,\mathbf{r})$ is the complex frequency response (amplitude and phase) at position $\mathbf{r}=(x,y,z)$ of the transmit aperture, also known as the *complex transmit aperture function*, with units of inverse-seconds per volt, where f represents *input* or *transmitted* frequencies in hertz. Equation (2.2-1) represents the *coupling* of the input electrical signal to the fluid medium, that is, the production of the input acoustic signal (source distribution) due to the input electrical signal via the transmit aperture whose performance is described, in part, by the complex transmit aperture function. Another useful result can be obtained from (2.2-1) as follows. Since

$$x_M(t,\mathbf{r}) = F_f^{-1}\left\{X(f,\mathbf{r}) A_T(f,\mathbf{r})\right\}, \tag{2.2-3}$$

then

$$\begin{aligned} X_M(f,\mathbf{r}) &= F_t\{x_M(t,\mathbf{r})\} = \int_{-\infty}^{\infty} x_M(t,\mathbf{r})\exp(-j2\pi ft)\,dt \\ &= X(f,\mathbf{r}) A_T(f,\mathbf{r}), \end{aligned} \tag{2.2-4}$$

or,

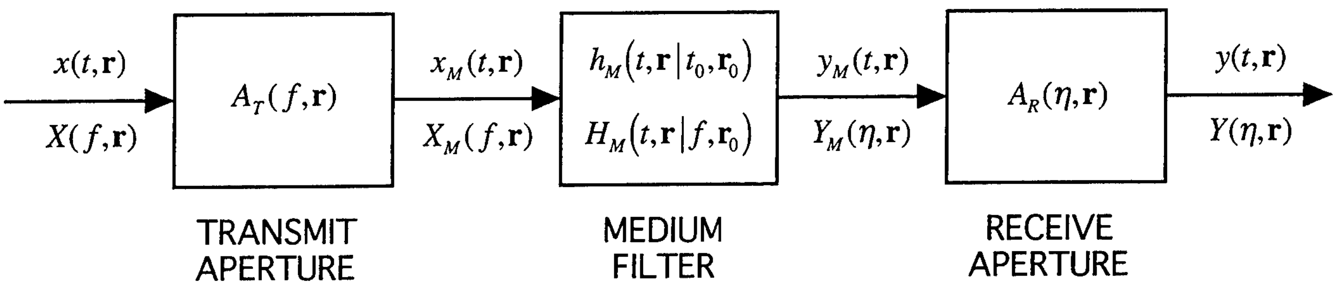

Figure 2.2-1. Block diagram model of small-amplitude pulse propagation.

$$\boxed{X_M(f,\mathbf{r}) = X(f,\mathbf{r})A_T(f,\mathbf{r}),} \tag{2.2-5}$$

where $X_M(f,\mathbf{r})$ is the complex frequency spectrum of the input acoustic signal (source distribution) at position $\mathbf{r}=(x,y,z)$ of the transmit aperture with units of inverse-seconds per hertz (see [9] and Fig. 2.2-1).

The second coupling equation is the *receive-aperture coupling equation* and is given by (see [9] and Fig. 2.2-1)

$$\boxed{y(t,\mathbf{r}) = \int_{-\infty}^{\infty} Y_M(\eta,\mathbf{r})A_R(\eta,\mathbf{r})\exp(+j2\pi\eta t)\,d\eta,} \tag{2.2-6}$$

where $y(t,\mathbf{r})$ is the output electrical signal from the receive aperture at time t and position $\mathbf{r}=(x,y,z)$ with units of volts,

$$Y_M(\eta,\mathbf{r}) = F_t\{y_M(t,\mathbf{r})\} = \int_{-\infty}^{\infty} y_M(t,\mathbf{r})\exp(-j2\pi\eta t)\,dt \tag{2.2-7}$$

is the complex frequency spectrum of the acoustic signal (velocity potential) incident upon the receive aperture at position $\mathbf{r}=(x,y,z)$ with units of $\left(\mathrm{m}^2/\mathrm{sec}\right)/\mathrm{Hz}$, $y_M(t,\mathbf{r})$ is the acoustic signal (velocity potential) incident upon the receive aperture at time t and position $\mathbf{r}=(x,y,z)$ with units of squared-meters per second, and $A_R(\eta,\mathbf{r})$ is the complex frequency response (amplitude and phase) at position $\mathbf{r}=(x,y,z)$ of the receive aperture, also known as the *complex receive aperture function*, with units of $\mathrm{V}/\left(\mathrm{m}^2/\mathrm{sec}\right)$, where η represents *output* or *received* frequencies in hertz. Note that in general, $\eta \neq f$. Equation (2.2-6) represents the *coupling* of the fluid medium to the output electrical signal, that is, the production of the output electrical signal due to the input acoustic signal via the receive aperture whose performance is described, in part, by the complex receive aperture function. Another useful result can be obtained from (2.2-6) as follows. Since

$$y(t,\mathbf{r}) = F_\eta^{-1}\{Y_M(\eta,\mathbf{r})A_R(\eta,\mathbf{r})\}, \tag{2.2-8}$$

then

$$\begin{aligned} Y(\eta,\mathbf{r}) &= F_t\{y(t,\mathbf{r})\} = \int_{-\infty}^{\infty} y(t,\mathbf{r})\exp(-j2\pi\eta t)\,dt \\ &= Y_M(\eta,\mathbf{r})A_R(\eta,\mathbf{r}), \end{aligned} \tag{2.2-9}$$

or,

$$\boxed{Y(\eta,\mathbf{r}) = Y_M(\eta,\mathbf{r})A_R(\eta,\mathbf{r}),} \tag{2.2-10}$$

where $Y(\eta,\mathbf{r})$ is the complex frequency spectrum of the output electrical signal at position $\mathbf{r}=(x,y,z)$ of the receive aperture with units of volts per hertz (see [9] and Fig. 2.2-1).

Now that we have introduced the two coupling equations, we will next show how to use them to solve the pulse-propagation problem. Recall from Section 2.1 that $x_M(t,\mathbf{r})$ and $y_M(t,\mathbf{r})$ are related by the following *linear*, three-dimensional, inhomogeneous wave equation [see (2.1-1)]:

$$\nabla^2 y_M(t,\mathbf{r}) - \frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2} y_M(t,\mathbf{r}) = x_M(t,\mathbf{r}). \tag{2.2-11}$$

Also recall from Section 2.1 that the solution of (2.2-11) is given by [see (2.1-2)]

$$y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_M(t_0,\mathbf{r}_0) h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0)\, dt_0\, d\mathbf{r}_0 , \tag{2.2-12}$$

where $h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0)$ is the *time-variant, space-variant, impulse response* (*Green's function*) of the medium filter (see Fig. 2.2-1), that is, it is the response of the filter at time t and position $\mathbf{r} = (x,y,z)$ due to the application of a unit-amplitude impulse at time t_0 and position $\mathbf{r}_0 = (x_0, y_0, z_0)$. Substituting the first coupling equation given by (2.2-1) into (2.2-12) and making use of (2.1-4) yields

$$\boxed{y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f,\mathbf{r}_0) A_T(f,\mathbf{r}_0) H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0)\, d\mathbf{r}_0 \exp(+j2\pi f t)\, df,} \tag{2.2 - 13}$$

where

$$H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0) \triangleq \int_{-\infty}^{\infty} h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0) \exp[-j2\pi f(t-t_0)]\, dt_0 \tag{2.2-14}$$

is the *time-variant, space-variant, complex frequency response* of the ocean at frequency f hertz. [see (2.1-4) and Fig. 2.2-1]. Note that since the impulse response of the ocean medium $h_M(t,\mathbf{r}\,|\,t_0,\mathbf{r}_0)$ is a *real* function,

$$\boxed{H_M^*(t,\mathbf{r}\,|\,-f,\mathbf{r}_0) = H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0).} \tag{2.2 - 15}$$

Equation (2.2-13) is a general solution for the acoustic signal (velocity potential) that is incident upon the receive aperture. It can be used in conjunction with (2.2-6) and (2.2-7) to solve for the output electrical signal from the receive aperture. Before we discuss how to simplify (2.2-13) in order to solve real-world, pulse-propagation problems, let us make the following interesting observation. If we refer back to the time-harmonic solution given by (2.1-10), and if we set $x_{f,M}(\boldsymbol{\zeta}) = X(f,\boldsymbol{\zeta}) A_T(f,\boldsymbol{\zeta})$, then (2.2-13) can be interpreted as being equal to the integration of the time-harmonic solution given by (2.1-10) over all the frequency components contained in the complex frequency spectrum of the input electrical signal applied to the transmit aperture.

We begin the simplification of (2.2-13) by assuming that an *identical* input electrical signal is applied at *all* spatial locations $\mathbf{r}_0$ of the transmit aperture, that is,

$$x(t,\mathbf{r}_0) = x(t) \tag{2.2-16}$$

so that [see (2.2-2)]

$$X(f,\mathbf{r}_0) = X(f). \tag{2.2-17}$$

Equations (2.2-16) and (2.2-17) are, in fact, *valid* in the vast majority of real-world, practical problems. For example, in most practical situations, an *identical* input electrical signal is applied to *all* elements in a transmit array *before* the application of complex weights for beamforming purposes. Substituting (2.2-17) into (2.2-13) yields

$$y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty} X(f) \int_{-\infty}^{\infty} A_T(f,\mathbf{r}_0) H_M(t,\mathbf{r}\,|\,f,\mathbf{r}_0)\, d\mathbf{r}_0 \exp(+j2\pi ft)\, df. \tag{2.2-18}$$

Equation (2.2-18) can be simplified by specifying the transmit aperture. In this paper, both the transmit and receive apertures will be modeled as *volume, conformal arrays* composed of *unevenly-spaced, complex-weighted, point elements*. This type of model for both of the apertures will give us maximum flexibility. Therefore, the complex transmit aperture function is given by

$$A_T(f,\mathbf{r}_0) = \sum_{i=1}^{N_T} c_{T_i}(f) S_{T_i}(f) \delta(\mathbf{r}_0 - \mathbf{r}_{T_i}), \tag{2.2-19}$$

where N_T is the total number of elements in the transmit array,

$$c_{T_i}(f) = a_{T_i}(f) \exp\left[+j\theta_{T_i}(f)\right] \tag{2.2-20}$$

is the dimensionless, frequency-dependent *complex weight* associated with element i in the transmit array, where $a_{T_i}(f)$ and $\theta_{T_i}(f)$ are *real*, frequency-dependent, *amplitude and phase weights*, respectively, $S_{T_i}(f)$ is the frequency-dependent *transmitter sensitivity* of element i in the transmit array with units of $\left(\mathrm{m}^3/\mathrm{sec}\right)/\mathrm{V}$, $\mathbf{r}_{T_i} = (x_{T_i}, y_{T_i}, z_{T_i})$ is the position vector to element i in the transmit array, and the impulse function $\delta(\mathbf{r}_0 - \mathbf{r}_{T_i})$, with units of inverse cubic meters, represents a point element at $\mathbf{r}_{T_i} = (x_{T_i}, y_{T_i}, z_{T_i})$. The real amplitude weights are *dimensionless* and are *even functions of frequency*, that is,

$$a_{T_i}(-f) = a_{T_i}(f), \tag{2.2-21}$$

and the real phase weights, with units of radians, are *odd functions of frequency*, that is,

$$\theta_{T_i}(-f) = -\theta_{T_i}(f). \tag{2.2-22}$$

Therefore,

$$\boxed{c^*_{T_i}(-f) = c_{T_i}(f).} \tag{2.2-23}$$

In addition, since the point elements are electroacoustic transducers with *real* impulse-response functions,

$$\boxed{S^*_{T_i}(-f) = S_{T_i}(f).} \tag{2.2-24}$$

Because of (2.2-23) and (2.2-24),

$$\boxed{A_T^*(-f,\mathbf{r}_0) = A_T(f,\mathbf{r}_0).} \tag{2.2-25}$$

Substituting (2.2-19) into (2.2-18) yields

$$y_M(t,\mathbf{r}) = \int_{-\infty}^{\infty} X(f) \sum_{i=1}^{N_T} c_{T_i}(f) S_{T_i}(f) H_M\left(t,\mathbf{r} \,|\, f,\mathbf{r}_{T_i}\right) \exp(+j2\pi ft)\, df, \tag{2.2-26}$$

or

$$y_M(t,\mathbf{r}) = \sum_{i=1}^{N_T} \int_{-\infty}^{\infty} X(f) c_{T_i}(f) S_{T_i}(f) H_M\left(t,\mathbf{r} \,|\, f,\mathbf{r}_{T_i}\right) \exp(+j2\pi ft)\, df. \tag{2.2-27}$$

Equation (2.2-27) indicates that the acoustic signal (velocity potential) $y_M(t,\mathbf{r})$ incident upon the receive aperture at time t and position $\mathbf{r} = (x,y,z)$ is equal to the sum of the contributions from each element in the transmit array.

Equation (2.2-27) can be simplified by specifying the class of transmitted electrical signals to be considered. Typical transmitted electrical signals used in active radar and sonar systems can be modeled as *amplitude-and-angle-modulated carriers* as follows [10]:

$$x(t) = A\,a(t)\cos[2\pi f_c t + \theta(t)]\,\mathrm{rect}\left(\frac{t-0.5T}{T}\right), \tag{2.2-28}$$

where A is the *amplitude scaling factor* with units of volts, $a(t)$ is a *dimensionless amplitude-modulating signal* where $|a(t)| \le 1$, f_c is the *carrier frequency* in hertz, $\theta(t)$ is an *angle-modulating signal* (also known as the *phase deviation*) with units of radians, $\cos(2\pi f_c t)$ is the *carrier waveform*, and

$$\mathrm{rect}\left(\frac{t-0.5T}{T}\right) = \begin{cases} 1, & 0 \le t \le T, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2-29}$$

is the *rectangle function* where T is the *pulse length* in seconds. The corresponding *complex envelope* of $x(t)$, denoted by $\tilde{x}(t)$, is given by [10]

$$\tilde{x}(t) = A\,a(t)\exp[+j\theta(t)]\,\mathrm{rect}\left(\frac{t-0.5T}{T}\right), \tag{2.2-30}$$

provided that $f_c > W$, where W is the bandwidth of $\tilde{x}(t)$ in hertz. The amplitude-and-angle-modulated carrier $x(t)$ is a *real*, *bandpass* signal whose magnitude spectrum $|X(f)|$ is centered about $f = \pm f_c$ hertz, where

$$X(f) = F_t\{x(t)\} = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\, dt. \tag{2.2-31}$$

However, its complex envelope $\tilde{x}(t)$ is a *complex*, *baseband* signal whose magnitude spectrum $\left|\tilde{X}(f)\right|$ is centered about $f = 0$ hertz, where

$$\tilde{X}(f) = F_t\{\tilde{x}(t)\} = \int_{-\infty}^{\infty} \tilde{x}(t)\exp(-j2\pi ft)\,dt. \tag{2.2-32}$$

The relationship between $X(f)$ and $\tilde{X}(f)$ is given by [11]

$$\boxed{X(f) = 0.5\left[\tilde{X}(f - f_c) + \tilde{X}^*\left(-[f + f_c]\right)\right].} \tag{2.2-33}$$

Substituting (2.2-33) into (2.2-27) and making use of (2.2-15), (2.2-23), and (2.2-24) yields

$$y_M(t,\mathbf{r}) = \operatorname{Re}\left\{\tilde{y}_M(t,\mathbf{r})\exp(+j2\pi f_c t)\right\}, \tag{2.2-34}$$

where

$$\tilde{y}_M(t,\mathbf{r}) = \sum_{i=1}^{N_T} \int_{-\infty}^{\infty} \tilde{X}(f) c_{T_i}(f + f_c) S_{T_i}(f + f_c) H_M\left(t,\mathbf{r}\,|\,f + f_c,\mathbf{r}_{T_i}\right)\exp(+j2\pi ft)\,df \tag{2.2-35}$$

is the complex envelope of $y_M(t,\mathbf{r})$. Equations (2.2-34) and (2.2-35) describe the acoustic signal (velocity potential) $y_M(t,\mathbf{r})$ that is incident upon the receive aperture at time t and position $\mathbf{r} = (x,y,z)$ in terms of its complex envelope $\tilde{y}_M(t,\mathbf{r})$. What we need to do next is to turn our attention to the receive aperture itself so that we can predict the output electrical signal from the receive aperture.

Recall that both the transmit and receive apertures are being modeled as *volume*, *conformal arrays* composed of *unevenly-spaced*, *complex-weighted*, *point elements*. Therefore, the complex receive aperture function is given by

$$A_R(\eta,\mathbf{r}) = \sum_{n=1}^{N_R} c_{R_n}(\eta) S_{R_n}(\eta)\delta(\mathbf{r} - \mathbf{r}_{R_n}), \tag{2.2-36}$$

where N_R is the total number of elements in the receive array,

$$c_{R_n}(\eta) = a_{R_n}(\eta)\exp\left[+j\theta_{R_n}(\eta)\right] \tag{2.2-37}$$

is the dimensionless, frequency-dependent *complex weight* associated with element n in the receive array, where $a_{R_n}(\eta)$ and $\theta_{R_n}(\eta)$ are *real*, frequency-dependent, *amplitude and phase weights*, respectively, $S_{R_n}(\eta)$ is the frequency-dependent *receiver sensitivity* of element n in the receive array with units of $\left(\mathrm{V}\text{-}\mathrm{m}^3\right)/\left(\mathrm{m}^2/\mathrm{sec}\right)$, or $V/\left(\left(\mathrm{m}^2/\mathrm{sec}\right)/\mathrm{m}^3\right)$, $\mathbf{r}_{R_n} = (x_{R_n}, y_{R_n}, z_{R_n})$ is the position vector to element n in the receive array, and the impulse function $\delta(\mathbf{r} - \mathbf{r}_{R_n})$, with units of inverse cubic meters, represents a point element at $\mathbf{r}_{R_n} = (x_{R_n}, y_{R_n}, z_{R_n})$. The real amplitude weights are *dimensionless* and are *even functions of frequency*, that is,

$$a_{R_n}(-\eta) = a_{R_n}(\eta), \tag{2.2-38}$$

and the real phase weights, with units of radians, are *odd functions of frequency*, that is,

$$\theta_{R_n}(-\eta) = -\theta_{R_n}(\eta). \tag{2.2-39}$$

Therefore,

$$\boxed{c^*_{R_n}(-\eta) = c_{R_n}(\eta).} \tag{2.2-40}$$

In addition, since the point elements are electroacoustic transducers with *real* impulse-response functions,

$$\boxed{S^*_{R_n}(-\eta) = S_{R_n}(\eta).} \tag{2.2-41}$$

Because of (2.2-40) and (2.2-41),

$$\boxed{A^*_R(-\eta,\mathbf{r}) = A_R(\eta,\mathbf{r}).} \tag{2.2-42}$$

Substituting (2.2-36) into (2.2-6) yields

$$y(t,\mathbf{r}) = \sum_{n=1}^{N_R} \int_{-\infty}^{\infty} Y_M(\eta,\mathbf{r}) c_{R_n}(\eta) S_{R_n}(\eta) \exp(+j2\pi\eta t)\, d\eta\, \delta(\mathbf{r}-\mathbf{r}_{R_n}), \tag{2.2-43}$$

or

$$\boxed{y(t,\mathbf{r}_{R_n}) = \int_{-\infty}^{\infty} Y_M(\eta,\mathbf{r}_{R_n}) c_{R_n}(\eta) S_{R_n}(\eta) \exp(+j2\pi\eta t)\, d\eta, \qquad n=1,2,\ldots,N_R,}$$

(2.2-44)

where $y(t,\mathbf{r}_{R_n})$ is the output electrical signal from element n in the receive array at time t and position $\mathbf{r}_{R_n} = (x_{R_n}, y_{R_n}, z_{R_n})$ with units of volts,

$$\boxed{Y_M(\eta,\mathbf{r}_{R_n}) = F_t\{y_M(t,\mathbf{r}_{R_n})\} = \int_{-\infty}^{\infty} y_M(t,\mathbf{r}_{R_n}) \exp(-j2\pi\eta t)\, dt} \tag{2.2-45}$$

is the complex frequency spectrum of the acoustic signal (velocity potential) incident upon element n in the receive array at η hertz and position $\mathbf{r}_{R_n} = (x_{R_n}, y_{R_n}, z_{R_n})$ with units of $(\text{m}^2/\text{sec})/\text{Hz}$ [see (2.2-7)],

$$\boxed{y_M(t,\mathbf{r}_{R_n}) = \mathrm{Re}\left\{\tilde{y}_M(t,\mathbf{r}_{R_n})\exp(+j2\pi f_c t)\right\}} \tag{2.2-46}$$

is the acoustic signal (velocity potential) incident upon element n in the receive array at time t and position $\mathbf{r}_{R_n} = (x_{R_n}, y_{R_n}, z_{R_n})$ with units of squared-meters per second [see (2.2-34)], and

$$\boxed{\tilde{y}_M(t,\mathbf{r}_{R_n}) = \sum_{i=1}^{N_T} \int_{-\infty}^{\infty} \tilde{X}(f) c_{T_i}(f+f_c) S_{T_i}(f+f_c) H_M\left(t,\mathbf{r}_{R_n} \middle| f+f_c, \mathbf{r}_{T_i}\right)\exp(+j2\pi f t)\,df}$$

(2.2-47)

is the complex envelope of $y_M(t,\mathbf{r})$ [see (2.2-35)]. Equations (2.2-44) through (2.2-47) are the *pulse-propagation coupling equations*. They couple the output electrical signal at a point element in a receive array to the transmitted electrical signal at the input to a transmit array via the complex frequency response of a fluid medium (e.g., air or water). The pulse-propagation coupling equations are based on *linear, time-variant, space-variant, filter theory*, the principles of *complex aperture theory* and *array theory*, and solving the *linear wave equation*. They provide a consistent, logical, and straightforward mathematical framework for the solution of small-amplitude, acoustic, pulse-propagation problems. Examples on the use of the pulse-propagation coupling equations are given in Section 3.

3 Bistatic Scattering

3.1 No Motion

In this section we will analyze the bistatic scattering problem shown in Fig. 3.1-1. The transmitter, the discrete point scatterer, and the receiver are *not* in motion. For example purposes, they are placed in *deep water* so that the ocean medium can be treated as being *unbounded*, that is, there are *no* boundaries and, hence, *no* reflections. The speed of sound and ambient density are *constants*. Although the propagation of sound between the source and discrete point scatterer, and between the discrete point scatterer and receiver can be treated as transmission through linear, time-invariant, *space-invariant* filters; the overall solution for this bistatic scattering problem corresponds to transmission through a linear, time-invariant, *space-variant* filter. The presence of a discrete point scatterer in an unbounded, homogeneous fluid medium (i.e., a fluid medium with constant speed of sound and ambient density) causes the medium to be space-variant.

Let the source distribution $x_M(t,\mathbf{r})$ be a *motionless, time-harmonic, point source* with units of inverse seconds, that is, let [see (2.1-8) and (2.1-11)]

$$x_M(t,\mathbf{r}) = S_0\delta(\mathbf{r}-\mathbf{r}_0)\exp(+j2\pi f t), \tag{3.1-1}$$

where S_0 is the *source strength* in cubic meters per second and the impulse function $\delta(\mathbf{r}-\mathbf{r}_0)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_0 = (x_0, y_0, z_0)$. The propagation of sound between the source and discrete point scatterer can be modeled as transmission through a linear, *time-invariant*, *space-invariant*, filter. Therefore, the acoustic field (velocity potential) incident upon the discrete point scatterer at $\mathbf{r}_1 = (x_1, y_1, z_1)$ is given by [see (2.1-17), (2.1-20), and (2.1-22)]

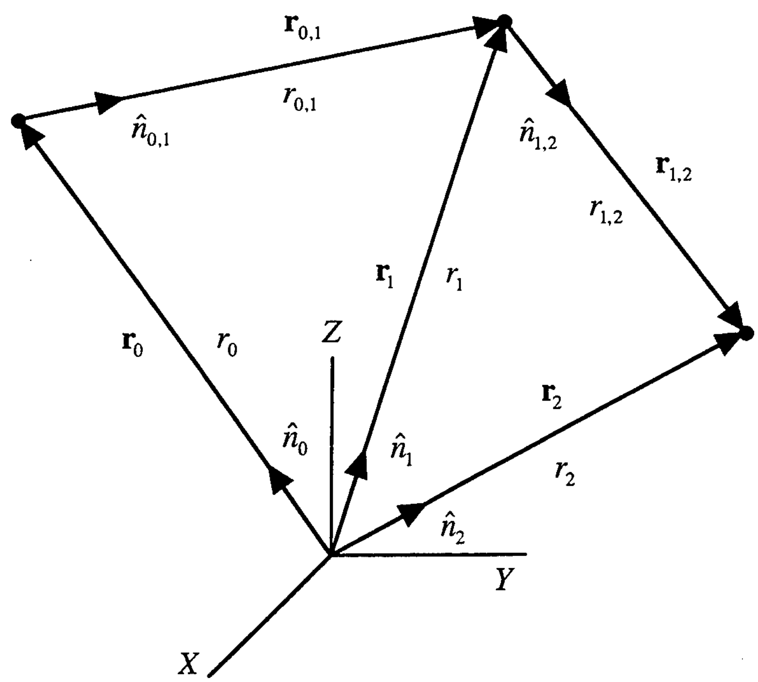

Figure 3.1-1. Bistatic scattering geometry. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. None of the platforms are in motion.

$$y_M(t,\mathbf{r}_1) = y_{f,M}(\mathbf{r}_1)\exp(+j2\pi ft), \tag{3.1-2}$$

where

$$y_{f,M}(\mathbf{r}_1) = S_0 H_M(f,\mathbf{r}_1 - \mathbf{r}_0) = -S_0 \frac{\exp\left(-jk\left|\mathbf{r}_1 - \mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}_1 - \mathbf{r}_0\right|}. \tag{3.1-3}$$

As can be seen from Fig. 3.1-1, the position vector from the point source to the discrete point scatterer is given by

$$\mathbf{r}_{0,1} = \mathbf{r}_1 - \mathbf{r}_0. \tag{3.1-4}$$

Therefore, (3.1-3) can be rewritten as

$$y_{f,M}(\mathbf{r}_1) = S_0 H_M(f,\mathbf{r}_{0,1}) = -S_0 \frac{\exp\left(-jk\left|\mathbf{r}_{0,1}\right|\right)}{4\pi\left|\mathbf{r}_{0,1}\right|}. \tag{3.1-5}$$

In order to compute the acoustic signal incident upon the receiver, we treat the discrete point scatterer as another *motionless*, *time-harmonic*, *point source* with units of inverse seconds, that is, let [see (3.1-1) and Fig. 3.1-2]

$$x'_M(t,\mathbf{r}) = S'_0\delta(\mathbf{r} - \mathbf{r}_1)\exp(+j2\pi ft), \tag{3.1-6}$$

where S'_0 is the *source strength* in cubic meters per second and the impulse function $\delta(\mathbf{r} - \mathbf{r}_1)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_1 = (x_1, y_1, z_1)$. The source strength S'_0 is given by

$$S'_0 = y_{f,M}(\mathbf{r}_1) g_1\left(f, \hat{n}_{0,1}, \hat{n}_{1,2}\right), \tag{3.1-7}$$

where $y_{f,M}(\mathbf{r}_1)$ is the spatial-dependent part of the time-harmonic velocity potential at $\mathbf{r}_1$ with units of squared-meters per second, given by either (3.1-3) or (3.1-5), and $g_1\left(f, \hat{n}_{0,1}, \hat{n}_{1,2}\right)$ is the *scattering amplitude function* of the discrete point scatterer with units of meters. The unit of meters for the scattering amplitude function represents an *effective scattering length* that may be larger or smaller than the actual length of the scatterer. The scattering amplitude function is a complex function (magnitude and phase) and is, in general, a function of frequency, the direction of wave propagation from the source to the scatterer ($\hat{n}_{0,1}$), and the direction of wave propagation from the scatterer to the receiver ($\hat{n}_{1,2}$) (see either Fig. 3.1-1 or Fig. 3.1-2) [12].

The propagation of sound between the discrete point scatterer and receiver can also be modeled as transmission through a linear, *time-invariant*, *space-invariant* filter. Therefore, the acoustic field (velocity potential) incident upon the receiver at $\mathbf{r}_2 = (x_2, y_2, z_2)$ due to a point source at $\mathbf{r}_1 = (x_1, y_1, z_1)$ is given by [see (2.1-17), (2.1-20), and (2.1-22)]

$$y_M(t,\mathbf{r}_2) = y_{f,M}(\mathbf{r}_2)\exp(+j2\pi ft), \tag{3.1-8}$$

where

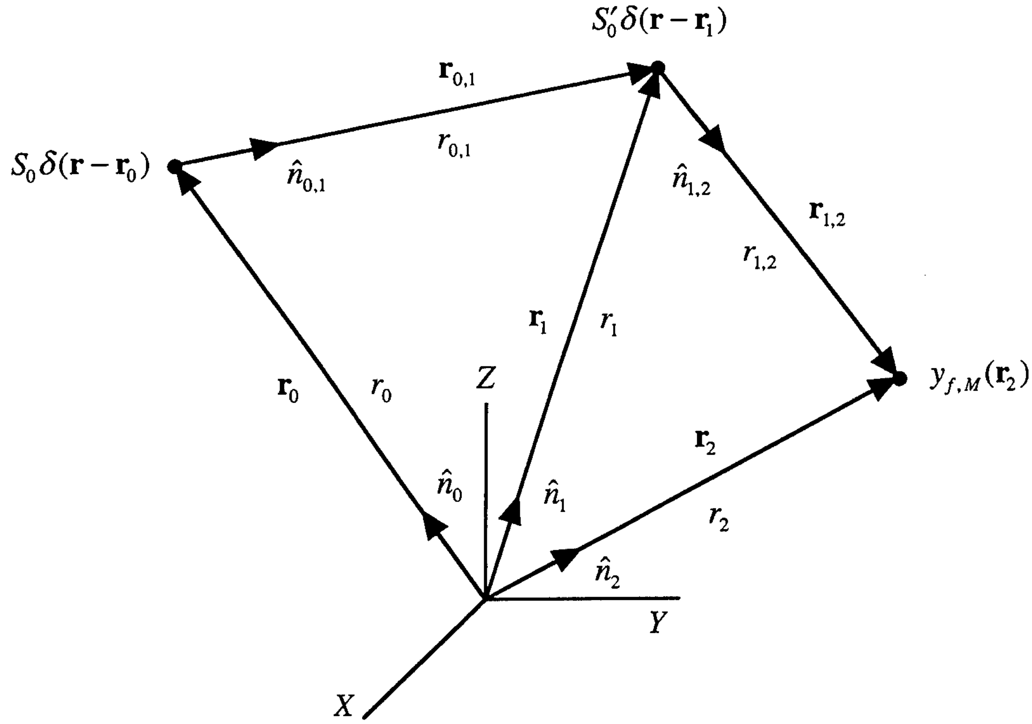

Figure 3.1-2. Bistatic scattering geometry. Both the transmitter at point 0, $P_0(\mathbf{r}_0)$, and the discrete point scatterer at point 1, $P_1(\mathbf{r}_1)$, are time-harmonic point sources. None of the platforms are in motion.

$$y_{f,M}(\mathbf{r}_2) = S_0' H_M(f,\mathbf{r}_2 - \mathbf{r}_1) = -S_0' \frac{\exp(-jk|\mathbf{r}_2 - \mathbf{r}_1|)}{4\pi|\mathbf{r}_2 - \mathbf{r}_1|}. \tag{3.1-9}$$

As can be seen from either Fig. 3.1-1 or Fig. 3.1-2, the position vector from the discrete point scatterer to the receiver is given by

$$\mathbf{r}_{1,2} = \mathbf{r}_2 - \mathbf{r}_1. \tag{3.1-10}$$

Therefore, (3.1-9) can be rewritten as

$$y_{f,M}(\mathbf{r}_2) = S_0' H_M(f,\mathbf{r}_{1,2}) = -S_0' \frac{\exp(-jk|\mathbf{r}_{1,2}|)}{4\pi|\mathbf{r}_{1,2}|}. \tag{3.1-11}$$

Let us now begin the process of obtaining final expressions for both the time-harmonic velocity potential and acoustic pressure incident upon the receiver. Substituting (3.1-7) and (3.1-5) into (3.1-11) yields

$$y_{f,M}(\mathbf{r}_2) = S_0 H_M(f,\mathbf{r}_{0,1}) g_1(f,\hat{n}_{0,1},\hat{n}_{1,2}) H_M(f,\mathbf{r}_{1,2}), \tag{3.1-12}$$

or, equivalently,

$$y_{f,M}(\mathbf{r}_2) = S_0 H_M(f,\mathbf{r}_2 \,|\, \mathbf{r}_0), \tag{3.1-13}$$

where

$$\begin{aligned} H_M(f,\mathbf{r}_2 \,|\, \mathbf{r}_0) &= H_M(f,\mathbf{r}_{0,1}) g_1(f,\hat{n}_{0,1},\hat{n}_{1,2}) H_M(f,\mathbf{r}_{1,2}) \\ &= g_1(f,\hat{n}_{0,1},\hat{n}_{1,2}) \frac{\exp[-jk(|\mathbf{r}_{0,1}| + |\mathbf{r}_{1,2}|)]}{16\pi^2|\mathbf{r}_{0,1}||\mathbf{r}_{1,2}|} \\ &= g_1(f,\hat{n}_{0,1},\hat{n}_{1,2}) \frac{\exp[-jk(|\mathbf{r}_1 - \mathbf{r}_0| + |\mathbf{r}_2 - \mathbf{r}_1|)]}{16\pi^2|\mathbf{r}_1 - \mathbf{r}_0||\mathbf{r}_2 - \mathbf{r}_1|}. \end{aligned} \tag{3.1-14}$$

Note that if the bistatic scattering problem shown in Fig. 3.1-1 corresponded to transmission through a space-invariant filter, then the complex frequency response given by (3.1-14) would be a function of the vector spatial difference $\mathbf{r}_2 - \mathbf{r}_0$, which it is *not*. Also note that in order to model the effects of frequency-dependent attenuation, simply replace the real wavenumber k given by (2.1-23) with the following *complex wavenumber* K:

$$\boxed{K = k - j\alpha(f),} \tag{3.1-15}$$

where $\alpha(f)$ is the *real*, *frequency-dependent*, *attenuation coefficient* in nepers per meter. In addition to being real quantities, both k and $\alpha(f)$ are *positive*.

With the use of (2.1-23), (3.1-8), (3.1-13), and (3.1-14); and by replacing the real wavenumber k in (3.1-14) with the complex wavenumber K given by (3.1-15), we can summarize our results as follows: for the bistatic scattering problem shown in Fig. 3.1-1, the time-harmonic velocity potential in squared-meters per second incident upon the receiver at $\mathbf{r}_2 = (x_2, y_2, z_2)$, due to a time-harmonic point source at $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a discrete point scatterer at $\mathbf{r}_1 = (x_1, y_1, z_1)$, is given by

$$\boxed{y_M(t, \mathbf{r}_2) = S_0 H_M\left(f, \mathbf{r}_2 \,|\, \mathbf{r}_0\right) \exp(+j2\pi f t), \qquad t \geq \tau,} \tag{3.1-16}$$

where S_0 is the *source strength* in cubic meters per second,

$$\boxed{H_M\left(f, \mathbf{r}_2 \,|\, \mathbf{r}_0\right) = g_1\left(f, \hat{n}_{0,1}, \hat{n}_{1,2}\right) \frac{\exp\left[-\alpha(f)(r_{0,1} + r_{1,2})\right]}{16\pi^2 r_{0,1} r_{1,2}} \exp(-j2\pi f \tau)} \tag{3.1-17}$$

is the *time-invariant, space-variant, complex frequency response* of the ocean at frequency f hertz, $g_1\left(f, \hat{n}_{0,1}, \hat{n}_{1,2}\right)$ is the *scattering amplitude function* of the discrete point scatterer in meters,

$$\hat{n}_{0,1} = \frac{\mathbf{r}_{0,1}}{\left|\mathbf{r}_{0,1}\right|} = \frac{\mathbf{r}_1 - \mathbf{r}_0}{\left|\mathbf{r}_1 - \mathbf{r}_0\right|} \tag{3.1-18}$$

and

$$\hat{n}_{1,2} = \frac{\mathbf{r}_{1,2}}{\left|\mathbf{r}_{1,2}\right|} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{\left|\mathbf{r}_2 - \mathbf{r}_1\right|} \tag{3.1-19}$$

are the dimensionless unit vectors in the directions measured from the source to the discrete point scatterer, and from the discrete point scatterer to the receiver, respectively, $\alpha(f)$ is the *real, frequency-dependent, attenuation coefficient* in nepers per meter,

$$\boxed{\tau = \frac{r_{0,1} + r_{1,2}}{c}} \tag{3.1-20}$$

is the *time delay* in seconds (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver), and

$$r_{0,1} = \left|\mathbf{r}_{0,1}\right| = \left|\mathbf{r}_1 - \mathbf{r}_0\right| \tag{3.1-21}$$

and

$$r_{1,2} = \left|\mathbf{r}_{1,2}\right| = \left|\mathbf{r}_2 - \mathbf{r}_1\right| \tag{3.1-22}$$

are the distances (ranges) in meters measured from the source to the discrete point scatterer, and from the discrete point scatterer to the receiver, respectively.

The general relationship between the *acoustic pressure* $p(t,\mathbf{r})$ in pascals (Pa) and the velocity potential $y_M(t,\mathbf{r})$ in squared-meters per second is given by

$$\boxed{p(t,\mathbf{r}) = -\rho_0(\mathbf{r})\frac{\partial}{\partial t}y_M(t,\mathbf{r}),} \tag{3.1-23}$$

where $\rho_0(\mathbf{r})$ is the ambient (equilibrium) density of the ocean in kilograms per cubic meter, shown as a function of position $\mathbf{r} = (x, y, z)$. Note that $1\ \text{Pa} = 1\ \text{N/m}^2$ where $1\ \text{N} = 1\ \text{kg-m/sec}^2$. Therefore, upon substituting (3.1-16) into (3.1-23), and since the ambient density is constant in our problem, the acoustic pressure in pascals incident upon the receiver at $\mathbf{r}_2 = (x_2, y_2, z_2)$ is given by

$$\boxed{p(t,\mathbf{r}_2) = -j2\pi f\rho_0 S_0 H_M\left(f,\mathbf{r}_2\,|\,\mathbf{r}_0\right)\exp(+j2\pi ft), \qquad t \geq \tau,} \tag{3.1-24}$$

where $H_M\left(f,\mathbf{r}_2\,|\,\mathbf{r}_0\right)$ is given by (3.1-17). Note that (3.1-24) can be rewritten as

$$p(t,\mathbf{r}_2) = p_f(\mathbf{r}_2)\exp(+j2\pi ft), \qquad t \geq \tau, \tag{3.1-25}$$

where the spatial-dependent part of the time-harmonic acoustic pressure in pascals is given by

$$p_f(\mathbf{r}_2) = -j2\pi f\rho_0 S_0 H_M\left(f,\mathbf{r}_2\,|\,\mathbf{r}_0\right). \tag{3.1-26}$$

Let us next relate the scattering amplitude function, the differential scattering cross section, and target strength of the discrete point scatterer. The *target strength* (TS) is defined as follows [13]:

$$\boxed{\text{TS} \triangleq 10\log_{10}\left[\frac{\sigma_d\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right)}{A_{\text{ref}}}\right]\text{dB re } A_{\text{ref}},} \tag{3.1-27}$$

where [12, 13]

$$\boxed{\sigma_d\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right) \triangleq \lim_{r_{1,2}\to\infty}\left[\frac{r_{1,2}^2\, I_{avg_s}(\mathbf{r}_2)}{I_{avg_i}(\mathbf{r}_1)}\right] = \frac{\left|g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right)\right|^2}{(4\pi)^2}} \tag{3.1-28}$$

is the *differential scattering cross section* with units of squared meters, $I_{avg_i}(\mathbf{r}_1)$ and $I_{avg_s}(\mathbf{r}_2)$ are the *time-average*, *incident* and *scattered intensities*, respectively, with units of watts per squared meter, $g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right)$ is the *scattering amplitude function* of the discrete point scatterer with units of meters, and A_{ref} is a *reference cross-sectional area* commonly chosen to be equal to $1\ \text{m}^2$.

Example 3.1-1 Monostatic Scattering Geometry

For a *monostatic* (*backscatter*) scattering geometry, both the transmitter and receiver are located at the same position, that is,

$$\mathbf{r}_2 = \mathbf{r}_0. \tag{3.1-29}$$

Therefore (see Fig. 3.1-1),

$$r_{1,2} = r_{0,1} \tag{3.1-30}$$

and

$$\hat{n}_{1,2} = -\hat{n}_{0,1}. \tag{3.1-31}$$

With the use of (3.1-30) and (3.1-31), the time-invariant, space-variant, complex frequency response of the ocean given by (3.1-17) reduces to

$$H_M\left(f,\mathbf{r}_2 \,|\, \mathbf{r}_0\right) = g_1\left(f,\hat{n}_{0,1},-\hat{n}_{0,1}\right)\frac{\exp\left[-2\alpha(f)r_{0,1}\right]}{(4\pi r_{0,1})^2}\exp(-j2\pi f\tau), \tag{3.1-32}$$

where $\hat{n}_{0,1}$ is given by (3.1-18),

$$\tau = \frac{2r_{0,1}}{c} \tag{3.1-33}$$

is the time delay in seconds, and $r_{0,1}$ is given by (3.1-21).

Example 3.1-2 Pulse Propagation

In this example we will demonstrate how to use the pulse-propagation coupling equations given by (2.2-44) through (2.2-47) for the bistatic scattering problem shown in Fig. 3.1-1. We begin by evaluating (2.2-47), which is repeated below for convenience:

$$\tilde{y}_M(t,\mathbf{r}_{R_n}) = \sum_{i=1}^{N_T}\int_{-\infty}^{\infty}\tilde{X}(f)c_{T_i}(f+f_c)S_{T_i}(f+f_c)H_M\left(t,\mathbf{r}_{R_n}\middle|f+f_c,\mathbf{r}_{T_i}\right)\exp(+j2\pi ft)\,df, \tag{2.2-47}$$

where $\tilde{y}_M(t,\mathbf{r}_{R_n})$ is the complex envelope of the acoustic signal (velocity potential) incident upon element n in the receive array at time t and position $\mathbf{r}_{R_n} = (x_{R_n}, y_{R_n}, z_{R_n})$ with units of squared-meters per second.

The complex frequency response of the ocean that describes the bistatic scattering problem shown in Fig. 3.1-1 is given by (3.1-17) and is time-invariant and space-variant. Therefore [see (2.1-29)],

$$H_M\left(t,\mathbf{r}\,|\,f,\mathbf{r}_0\right) = H_M\left(f,\mathbf{r}\,|\,\mathbf{r}_0\right), \tag{3.1-34}$$

and as a result,

$$H_M\left(t,\mathbf{r}_{R_n}\middle|f+f_c,\mathbf{r}_{T_i}\right)=H_M\left(f+f_c,\mathbf{r}_{R_n}\middle|\mathbf{r}_{T_i}\right). \tag{3.1-35}$$

Substituting (3.1-17) into (3.1-35) yields

$$H_M\left(t,\mathbf{r}_{R_n}\middle|f+f_c,\mathbf{r}_{T_i}\right)=g_1\left(f+f_c,\hat{n}_{T_i,1},\hat{n}_{1,R_n}\right)\frac{\exp\left[-\alpha(f+f_c)(r_{T_i,1}+r_{1,R_n})\right]}{16\pi^2 r_{T_i,1}r_{1,R_n}}\exp\left[-j2\pi(f+f_c)\tau_{T_i,R_n}\right], \tag{3.1-36}$$

where [see (3.1-18)]

$$\hat{n}_{T_i,1}=\frac{\mathbf{r}_{T_i,1}}{\left|\mathbf{r}_{T_i,1}\right|}=\frac{\mathbf{r}_1-\mathbf{r}_{T_i}}{\left|\mathbf{r}_1-\mathbf{r}_{T_i}\right|} \tag{3.1-37}$$

and [see (3.1-19)]

$$\hat{n}_{1,R_n}=\frac{\mathbf{r}_{1,R_n}}{\left|\mathbf{r}_{1,R_n}\right|}=\frac{\mathbf{r}_{R_n}-\mathbf{r}_1}{\left|\mathbf{r}_{R_n}-\mathbf{r}_1\right|} \tag{3.1-38}$$

are the dimensionless unit vectors in the directions measured from element i in the transmit array to the discrete point scatterer, and from the discrete point scatterer to element n in the receive array, respectively,

$$\tau_{T_i,R_n}=\frac{r_{T_i,1}+r_{1,R_n}}{c} \tag{3.1-39}$$

is the time delay in seconds (the amount of time it takes for the acoustic signal transmitted from element i in the transmit array to *begin* to appear at element n in the receive array) [see (3.1-20)], and [see (3.1-21)]

$$r_{T_i,1}=\left|\mathbf{r}_{T_i,1}\right|=\left|\mathbf{r}_1-\mathbf{r}_{T_i}\right| \tag{3.1-40}$$

and [see (3.1-22)]

$$r_{1,R_n}=\left|\mathbf{r}_{1,R_n}\right|=\left|\mathbf{r}_{R_n}-\mathbf{r}_1\right| \tag{3.1-41}$$

are the distances (ranges) in meters measured from element i in the transmit array to the discrete point scatterer, and from the discrete point scatterer to element n in the receive array, respectively.

Substituting (3.1-36) into (2.2-47) yields

$$\tilde{y}_M(t,\mathbf{r}_{R_n})=\sum_{i=1}^{N_T}\int_{-\infty}^{\infty}\tilde{X}'_{T_i,R_n}(f)\exp\left[+j2\pi f(t-\tau_{T_i,R_n})\right]df\exp(-j2\pi f_c\tau_{T_i,R_n}), \tag{3.1-42}$$

or

$$\tilde{y}_M(t,\mathbf{r}_{R_n}) = \sum_{i=1}^{N_T} \tilde{x}'_{T_i,R_n}(t-\tau_{T_i,R_n})\exp(-j2\pi f_c \tau_{T_i,R_n}), \tag{3.1-43}$$

where

$$\tilde{X}'_{T_i,R_n}(f) = \tilde{X}(f)c_{T_i}(f+f_c)S_{T_i}(f+f_c)g_1\left(f+f_c,\hat{n}_{T_i,1},\hat{n}_{1,R_n}\right)\frac{\exp\left[-\alpha(f+f_c)(r_{T_i,1}+r_{1,R_n})\right]}{16\pi^2 r_{T_i,1}r_{1,R_n}} \tag{3.1-44}$$

and

$$\tilde{x}'_{T_i,R_n}(t) = F_f^{-1}\left\{\tilde{X}'_{T_i,R_n}(f)\right\} = \int_{-\infty}^{\infty}\tilde{X}'_{T_i,R_n}(f)\exp(+j2\pi ft)\,df. \tag{3.1-45}$$

Equation (3.1-43) indicates that the complex envelope, $\tilde{y}_M(t,\mathbf{r}_{R_n})$, of the acoustic signal (velocity potential) incident upon element n in the receive array is equal to the sum of time-delayed, modified versions of the complex envelopes of the transmitted electrical signals from each element in the transmit array. Equation (3.1-44) shows how the frequency spectrum of the complex envelope of the transmitted electrical signal, $\tilde{X}(f)$, gets modified by various other important frequency-dependent functions in order to produce the complex envelope, $\tilde{y}_M(t,\mathbf{r}_{R_n})$, of the acoustic signal (velocity potential) incident upon element n in the receive array.

If we next substitute (3.1-43) into (2.2-46), then we obtain

$$y_M(t,\mathbf{r}_{R_n}) = \sum_{i=1}^{N_T} \mathrm{Re}\left\{\tilde{x}'_{T_i,R_n}(t-\tau_{T_i,R_n})\exp\left[+j2\pi f_c(t-\tau_{T_i,R_n})\right]\right\}. \tag{3.1-46}$$

Since $x(t)$ given by (2.2-28) and $\tilde{x}(t)$ given by (2.2-30) are related by

$$x(t) = \mathrm{Re}\left\{\tilde{x}(t)\exp(+j2\pi f_c t)\right\}, \tag{3.1-47}$$

if we let

$$x'_{T_i,R_n}(t) = \mathrm{Re}\left\{\tilde{x}'_{T_i,R_n}(t)\exp(+j2\pi f_c t)\right\}, \tag{3.1-48}$$

then (3.1-46) can be rewritten as

$$y_M(t,\mathbf{r}_{R_n}) = \sum_{i=1}^{N_T} x'_{T_i,R_n}(t-\tau_{T_i,R_n}). \tag{3.1-49}$$

Substituting (3.1-49) into (2.2-45) yields

$$Y_M(\eta,\mathbf{r}_{R_n}) = \sum_{i=1}^{N_T} X'_{T_i,R_n}(\eta)\exp(-j2\pi\eta\tau_{T_i,R_n}), \tag{3.1-50}$$

and upon substituting (3.1-50) into (2.2-44), we obtain

$$y(t,\mathbf{r}_{R_n}) = \sum_{i=1}^{N_T} \int_{-\infty}^{\infty} X''_{T_i,R_n}(\eta) \exp\left[+j2\pi\eta(t-\tau_{T_i,R_n})\right] d\eta, \qquad n=1,2,\ldots,N_R, \tag{3.1-51}$$

or, finally,

$$y(t,\mathbf{r}_{R_n}) = \sum_{i=1}^{N_T} x''_{T_i,R_n}(t-\tau_{T_i,R_n}), \qquad n=1,2,\ldots,N_R, \tag{3.1-52}$$

where

$$X''_{T_i,R_n}(\eta) = X'_{T_i,R_n}(\eta)\, c_{R_n}(\eta)\, S_{R_n}(\eta) \tag{3.1-53}$$

and

$$x''_{T_i,R_n}(t) = F_\eta^{-1}\left\{X''_{T_i,R_n}(\eta)\right\} = \int_{-\infty}^{\infty} X''_{T_i,R_n}(\eta) \exp(+j2\pi\eta t)\, d\eta. \tag{3.1-54}$$

Equation (3.1-52) indicates that the output electrical signal $y(t,\mathbf{r}_{R_n})$ from element n in the receive array at time t and position $\mathbf{r}_{R_n} = (x_{R_n}, y_{R_n}, z_{R_n})$ with units of volts is equal to the sum of time-delayed, modified versions of the transmitted electrical signals from each element in the transmit array.

3.2 Discrete Point Scatterer In Motion

In this section we will analyze the bistatic scattering problem shown in Fig. 3.2-1. The transmitter and receiver are *not* in motion. Only the discrete point scatterer is in motion. For example purposes, all three platforms are placed in *deep water* so that the ocean medium can be treated as being *unbounded*, that is, there are *no* boundaries and, hence, *no* reflections. The speed of sound and ambient density are *constants*. Although the propagation of sound between the source and discrete point scatterer, and between the discrete point scatterer and receiver can be treated as transmission through linear, time-variant, *space-invariant* filters; the overall solution for this bistatic scattering problem corresponds to transmission through a linear, time-variant, *space-variant* filter. The presence of a discrete point scatterer in an unbounded, homogeneous fluid medium (i.e., a fluid medium with constant speed of sound and ambient density) causes the medium to be space-variant.

Let the source distribution $x_M(t,\mathbf{r})$ be a *motionless, time-harmonic, point source* with units of inverse seconds, that is, let [see (2.1-8) and (2.1-11)]

$$x_M(t,\mathbf{r}) = S_0\,\delta(\mathbf{r}-\mathbf{r}_0)\exp(+j2\pi f t), \tag{3.2-1}$$

where S_0 is the *source strength* in cubic meters per second and the impulse function $\delta(\mathbf{r}-\mathbf{r}_0)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_0 = (x_0, y_0, z_0)$. The *velocity vector* of the discrete point scatterer, $\mathbf{V}_1$, is given by

$$\mathbf{V}_1 = V_1 \hat{n}_{V_1}, \tag{3.2-2}$$

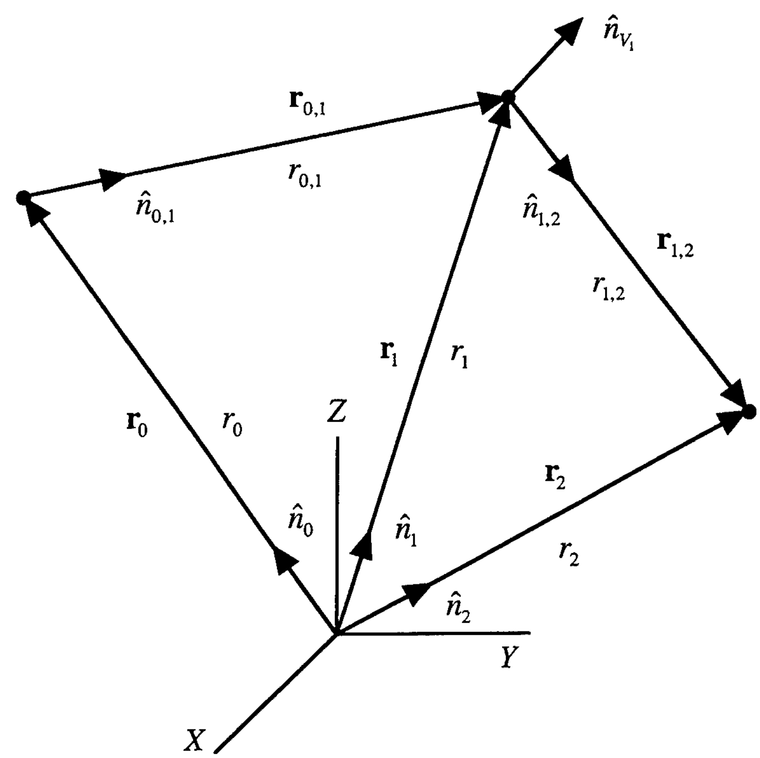

Figure 3.2-1. Bistatic scattering geometry when motion begins at time $t = t_0 = 0$ seconds. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. Only the discrete point scatterer is in motion.

where V_1 is the *speed* in meters per second and $\hat{n}_{V_1}$ is the dimensionless unit vector in the direction of $\mathbf{V}_1$. The velocity vector given by (3.2-2) is *constant*, that is, both the speed and direction are *constants* - there is *no* acceleration. Motion begins at time $t = t_0 = 0$ seconds.

Since the discrete point scatterer is now in motion, the position vector from the origin to the discrete point scatterer, denoted by $\boldsymbol{\mathcal{R}}_1(t)$, is a function of time given by

$$\boldsymbol{\mathcal{R}}_1(t) = \mathbf{r}_1 + \Delta t\,\mathbf{V}_1, \qquad t \ge 0, \tag{3.2-3}$$

where $\mathbf{r}_1 = (x_1, y_1, z_1)$ is the position vector from the origin to the discrete point scatterer when motion begins at time $t = t_0 = 0$ seconds (see Fig. 3.2-1), and

$$\begin{aligned} \Delta t &= t - t_0, \qquad & t &\ge t_0, \\ &= t, & t &\ge 0. \end{aligned} \tag{3.2-4}$$

Note that $\boldsymbol{\mathcal{R}}_1(0) = \mathbf{r}_1$.

When the transmitted acoustic field is first incident upon the discrete point scatterer at some time t' seconds where $t' > 0$, the position vector from the origin to the discrete point scatterer is given by [see (3.2-3) and Fig. 3.2-2]

$$\mathbf{r}_1' = \boldsymbol{\mathcal{R}}_1(t') = \mathbf{r}_1 + \Delta t'\,\mathbf{V}_1, \qquad t' > 0, \tag{3.2-5}$$

where

$$\begin{aligned} \Delta t' &= t' - t_0, \qquad & t' &> t_0, \\ &= t', & t' &> 0. \end{aligned} \tag{3.2-6}$$

The propagation of sound between the source and discrete point scatterer can be modeled as transmission through a linear, *time-variant*, *space-invariant* filter. Therefore, the acoustic field (velocity potential) incident upon the discrete point scatterer at time t' and position $\mathbf{r}_1' = (x_1', y_1', z_1')$ is given by [see (2.1-62)]

$$\begin{aligned} y_M(t', \mathbf{r}_1') &= S_0 H_M\left(t', \mathbf{r}_1' - \mathbf{r}_0 \,|\, f\right) \exp(+j2\pi f t') \\ &= -S_0 \frac{\exp\left(-jk\left|\mathbf{r}_1' - \mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}_1' - \mathbf{r}_0\right|} \exp(+j2\pi f t'), \qquad t' > 0, \end{aligned} \tag{3.2-7}$$

where

$$H_M\left(t', \mathbf{r}_1' - \mathbf{r}_0 \,|\, f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_1' - \mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}_1' - \mathbf{r}_0\right|}. \tag{3.2-8}$$

By referring to Fig. 3.2-2, we can express the position vector from the point source to the discrete point scatterer at time t' as

$$\mathbf{r}_{0,1}' = \mathbf{r}_1' - \mathbf{r}_0, \tag{3.2-9}$$

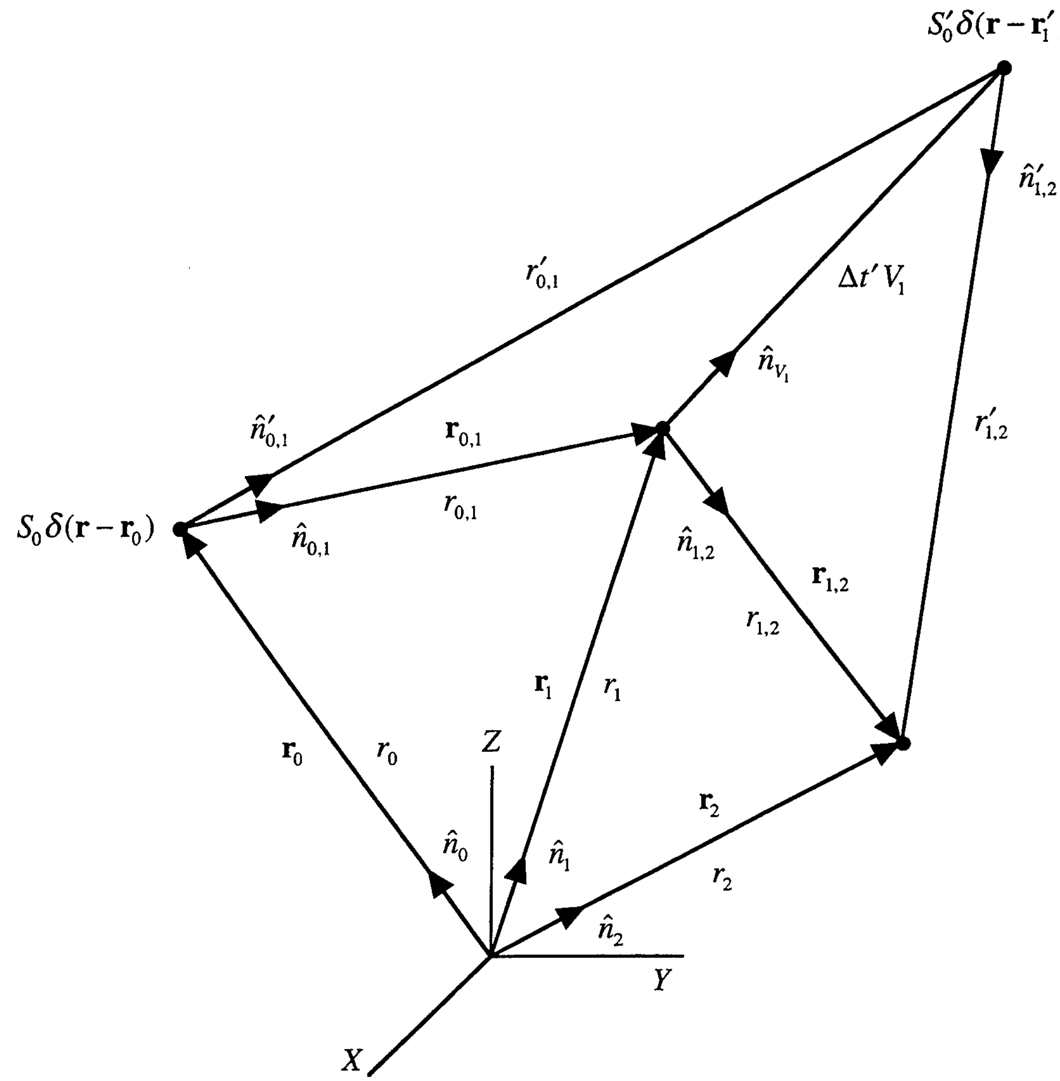

Figure 3.2-2. Bistatic scattering geometry when the transmitted acoustic field is first incident upon the discrete point scatterer at time t' seconds. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. Only the discrete point scatterer is in motion.

and upon substituting (3.2-5) into (3.2-9), we obtain

$$\mathbf{r}'_{0,1} = r'_{0,1}\hat{n}'_{0,1} = \mathbf{r}_{0,1} + \Delta t'\mathbf{V}_1, \tag{3.2-10}$$

where $\mathbf{r}_{0,1}$ is given by (3.1-4). Therefore, (3.2-7) and (3.2-8) can be rewritten as

$$\begin{aligned} y_M(t',\mathbf{r}'_1) &= S_0 H_M\left(t',\mathbf{r}'_{0,1}\middle|f\right)\exp(+j2\pi ft') \\ &= -S_0\frac{\exp\left(-jk\left|\mathbf{r}'_{0,1}\right|\right)}{4\pi\left|\mathbf{r}'_{0,1}\right|}\exp(+j2\pi ft'), \qquad t'>0, \end{aligned} \tag{3.2-11}$$

and

$$H_M\left(t',\mathbf{r}'_{0,1}\middle|f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}'_{0,1}\right|\right)}{4\pi\left|\mathbf{r}'_{0,1}\right|}. \tag{3.2-12}$$

In order to compute the acoustic signal incident upon the receiver, we treat the discrete point scatterer at time $t \geq t'$ and position $\mathbf{r}'_1$ as another *motionless*, *time-harmonic*, *point source* with units of inverse seconds, that is, let [see (3.2-1) and Fig. 3.2-2]

$$x'_M(t,\mathbf{r}) = S'_0\delta(\mathbf{r}-\mathbf{r}'_1)\exp(+j2\pi ft), \tag{3.2-13}$$

where S'_0 is the *source strength* in cubic meters per second and the impulse function $\delta(\mathbf{r}-\mathbf{r}'_1)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}'_1 = (x'_1, y'_1, z'_1)$. The source strength S'_0 is given by

$$S'_0 = S_0 H_M\left(t',\mathbf{r}'_{0,1}\middle|f\right)g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right), \tag{3.2-14}$$

where $g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)$ is the *scattering amplitude function* of the discrete point scatterer with units of meters as discussed in Section 3.1. The unit of meters for the scattering amplitude function represents an *effective scattering length* that may be larger or smaller than the actual length of the scatterer. As we discussed in Section 3.1, the scattering amplitude function is a complex function (magnitude and phase) and is, in general, a function of frequency, the direction of wave propagation from the source to the scatterer, and the direction of wave propagation from the scatterer to the receiver [12]. Since the discrete point scatterer is now in motion and the transmitted acoustic field is first incident upon the discrete point scatterer at time t', the direction of wave propagation from the source to the scatterer is given by the dimensionless unit vector $\hat{n}'_{0,1}$ (see Fig. 3.2-2). Similarly, since the discrete point scatterer is being treated as another point source at time t' and position $\mathbf{r}'_1$, the direction of wave propagation from the scatterer to the receiver is given by the dimensionless unit vector $\hat{n}'_{1,2}$ (see Fig. 3.2-2).

The propagation of sound between the discrete point scatterer and receiver can also be modeled as transmission through a linear, *time-variant*, *space-invariant* filter. The scattered acoustic field is first incident upon the receiver at some time t seconds where $t > t' > 0$. Therefore, the acoustic field (velocity potential) incident upon the receiver at time t and position $\mathbf{r}_2 = (x_2, y_2, z_2)$ due to a point source at $\mathbf{r}'_1 = (x'_1, y'_1, z'_1)$ is given by [see (2.1-62)]

$$y_M(t,\mathbf{r}_2) = S_0' H_M\left(t,\mathbf{r}_2 - \mathbf{r}_1' \,|\, f\right)\exp(+j2\pi ft)$$
$$= -S_0' \frac{\exp\left(-jk|\mathbf{r}_2 - \mathbf{r}_1'|\right)}{4\pi|\mathbf{r}_2 - \mathbf{r}_1'|}\exp(+j2\pi ft), \qquad t > t' > 0, \tag{3.2-15}$$

where

$$H_M\left(t,\mathbf{r}_2 - \mathbf{r}_1' \,|\, f\right) = -\frac{\exp\left(-jk|\mathbf{r}_2 - \mathbf{r}_1'|\right)}{4\pi|\mathbf{r}_2 - \mathbf{r}_1'|}. \tag{3.2-16}$$

By referring to Fig. 3.2-2, we can express the position vector from the discrete point scatterer to the receiver at time t' as

$$\mathbf{r}_{1,2}' = \mathbf{r}_2 - \mathbf{r}_1', \tag{3.2-17}$$

and upon substituting (3.2-5) into (3.2-17), we obtain

$$\mathbf{r}_{1,2}' = r_{1,2}'\hat{n}_{1,2}' = \mathbf{r}_{1,2} - \Delta t'\mathbf{V}_1, \tag{3.2-18}$$

where $\mathbf{r}_{1,2}$ is given by (3.1-10). Therefore, (3.2-15) and (3.2-16) can be rewritten as

$$y_M(t,\mathbf{r}_2) = S_0' H_M\left(t,\mathbf{r}_{1,2}' \,|\, f\right)\exp(+j2\pi ft)$$
$$= -S_0' \frac{\exp\left(-jk|\mathbf{r}_{1,2}'|\right)}{4\pi|\mathbf{r}_{1,2}'|}\exp(+j2\pi ft), \qquad t > t' > 0, \tag{3.2-19}$$

and

$$H_M\left(t,\mathbf{r}_{1,2}' \,|\, f\right) = -\frac{\exp\left(-jk|\mathbf{r}_{1,2}'|\right)}{4\pi|\mathbf{r}_{1,2}'|}. \tag{3.2-20}$$

Let us now begin the process of obtaining final expressions for both the time-harmonic velocity potential and acoustic pressure incident upon the receiver. Substituting (3.2-14) into (3.2-19) yields

$$y_M(t,\mathbf{r}_2) = S_0 H_M\left(t',\mathbf{r}_{0,1}' \,|\, f\right) g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}'\right) H_M\left(t,\mathbf{r}_{1,2}' \,|\, f\right)\exp(+j2\pi ft), \qquad t > t' > 0, \tag{3.2-21}$$

or, equivalently,

$$y_M(t,\mathbf{r}_2) = S_0 H_M\left(t,\mathbf{r}_2 \,|\, f,\mathbf{r}_0\right)\exp(+j2\pi ft), \qquad t > t' > 0, \tag{3.2-22}$$

where

$$
\begin{aligned}
H_M(t,\mathbf{r}_2\,|f,\mathbf{r}_0) &= H_M\left(t',\mathbf{r}'_{0,1}\middle|f\right)g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)H_M\left(t,\mathbf{r}'_{1,2}\middle|f\right) \\
&= g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)\frac{\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right]}{16\pi^2\left|\mathbf{r}'_{0,1}\right|\left|\mathbf{r}'_{1,2}\right|} \\
&= g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)\frac{\exp\left[-jk\left(\left|\mathbf{r}_1-\mathbf{r}_0+\Delta t'\mathbf{V}_1\right|+\left|\mathbf{r}_2-\mathbf{r}_1-\Delta t'\mathbf{V}_1\right|\right)\right]}{16\pi^2\left|\mathbf{r}_1-\mathbf{r}_0+\Delta t'\mathbf{V}_1\right|\left|\mathbf{r}_2-\mathbf{r}_1-\Delta t'\mathbf{V}_1\right|},
\end{aligned}
\tag{3.2-23}
$$

$\Delta t'$ is given by (3.2-6), and

$$
t = t' + \frac{\left|\mathbf{r}'_{1,2}\right|}{c}, \tag{3.2-24}
$$

or

$$
\boxed{t' = t - \frac{\left|\mathbf{r}'_{1,2}\right|}{c}.} \tag{3.2-25}
$$

Note that if the bistatic scattering problem shown in Fig. 3.2-1 corresponded to transmission through a space-invariant filter, then the complex frequency response given by (3.2-23) would be a function of the vector spatial difference $\mathbf{r}_2 - \mathbf{r}_0$, which it is *not*.

In order to simplify the complex frequency response given by (3.2-23), we need to derive approximate, simplified expressions for the ranges $\left|\mathbf{r}'_{0,1}\right|$ and $\left|\mathbf{r}'_{1,2}\right|$. Let us begin with $\left|\mathbf{r}'_{0,1}\right|$. Since

$$
\left|\mathbf{r}'_{0,1}\right|^2 = \mathbf{r}'_{0,1} \bullet \mathbf{r}'_{0,1}, \tag{3.2-26}
$$

substituting (3.2-10) into (3.2-26) yields

$$
\left|\mathbf{r}'_{0,1}\right|^2 = (\mathbf{r}_{0,1} + \Delta t'\mathbf{V}_1) \bullet (\mathbf{r}_{0,1} + \Delta t'\mathbf{V}_1). \tag{3.2-27}
$$

Expanding the right-hand side of (3.2-27) and taking the square root of both sides of the resulting equation yields

$$
\left|\mathbf{r}'_{0,1}\right| = r_{0,1}\left[1 + \frac{2}{r_{0,1}}(\hat{n}_{0,1} \bullet \mathbf{V}_1)\Delta t' + \left(\frac{V_1\Delta t'}{r_{0,1}}\right)^2\right]^{1/2}, \tag{3.2-28}
$$

or

$$
\left|\mathbf{r}'_{0,1}\right| = r_{0,1}\left\{1 + 2\frac{V_1\Delta t'}{r_{0,1}}\left[(\hat{n}_{0,1} \bullet \hat{n}_{V_1}) + \frac{1}{2}\frac{V_1\Delta t'}{r_{0,1}}\right]\right\}^{1/2}. \tag{3.2-29}
$$

Equations (3.2-28) and (3.2-29) are *exact* expressions for the range $|\mathbf{r}'_{0,1}|$. What we need to do next is to derive a valid binomial expansion of the square-root factor in (3.2-29) in order to simplify the expression for the range $|\mathbf{r}'_{0,1}|$. Equation (3.2-29) can be rewritten as

$$|\mathbf{r}'_{0,1}| = r_{0,1}\sqrt{1+b} \approx r_{0,1}\left(1+\frac{b}{2}-\frac{b^2}{8}+\cdots\right), \qquad |b|<1, \tag{3.2-30}$$

where

$$b = 2\frac{V_1\Delta t'}{r_{0,1}}\left[(\hat{n}_{0,1}\bullet\hat{n}_{V_1})+\frac{1}{2}\frac{V_1\Delta t'}{r_{0,1}}\right]. \tag{3.2-31}$$

Note that (3.2-30) is valid only if $|b|<1$, where b is given by (3.2-31). If we impose the more stringent criterion that

$$2\frac{V_1\Delta t'}{r_{0,1}} \le 0.1, \tag{3.2-32}$$

or

$$\boxed{\frac{V_1\Delta t'}{r_{0,1}} \le 0.05,} \tag{3.2-33}$$

then we can use only the first two terms in the binomial expansion. For example, since the maximum positive value of the dot product term in (3.2-31) is $+1$, and if (3.2-33) is satisfied, then

$$\begin{aligned} |b| &\le 0.1(1+0.025) \\ &\le 0.1025 \ll 1. \end{aligned} \tag{3.2-34}$$

Therefore, if (3.2-33) is satisfied, then we can use only the first two terms in the binomial expansion in (3.2-30). Doing so yields

$$|\mathbf{r}'_{0,1}| \approx r_{0,1} + V_1\Delta t'\left[(\hat{n}_{0,1}\bullet\hat{n}_{V_1})+\frac{1}{2}\frac{V_1\Delta t'}{r_{0,1}}\right]. \tag{3.2-35}$$

Equation (3.2-35) can be simplified further if we can justify neglecting the second term inside the square brackets. Consider the following criterion: the second term inside the square brackets in (3.2-35) can be neglected if

$$|\hat{n}_{0,1}\bullet\hat{n}_{V_1}| \ge 10\frac{1}{2}\frac{V_1\Delta t'}{r_{0,1}}, \tag{3.2-36}$$

or

$$\boxed{\left|\cos\alpha\right| \geq 5\frac{V_1 \Delta t'}{r_{0,1}},} \tag{3.2-37}$$

where α is the angle between the two unit vectors $\hat{n}_{0,1}$ and $\hat{n}_{V_1}$. The following two ranges of values for the angle α will satisfy (3.2-37):

$$\boxed{0° \leq \alpha \leq \cos^{-1}\left(5\frac{V_1 \Delta t'}{r_{0,1}}\right)} \tag{3.2-38}$$

and

$$\boxed{180° - \cos^{-1}\left(5\frac{V_1 \Delta t'}{r_{0,1}}\right) \leq \alpha \leq 180°.} \tag{3.2-39}$$

For example, if $V_1 \Delta t' / r_{0,1} = 0.05$ [see (3.2-33)], then $0° \leq \alpha \leq 75.5°$ and $104.5° \leq \alpha \leq 180°$ will satisfy (3.2-37). Therefore, if (3.2-37) is satisfied, then (3.2-35) reduces to

$$\boxed{\left|\mathbf{r}'_{0,1}\right| \approx r_{0,1} + (\hat{n}_{0,1} \bullet \mathbf{V}_1)\Delta t'.} \tag{3.2-40}$$

Equation (3.2-40) is our desired simplified expression for the range $\left|\mathbf{r}'_{0,1}\right|$. Let us next derive a similar expression for the range $\left|\mathbf{r}'_{1,2}\right|$.

Since

$$\left|\mathbf{r}'_{1,2}\right|^2 = \mathbf{r}'_{1,2} \bullet \mathbf{r}'_{1,2}, \tag{3.2-41}$$

substituting (3.2-18) into (3.2-41) yields

$$\left|\mathbf{r}'_{1,2}\right|^2 = (\mathbf{r}_{1,2} - \Delta t' \mathbf{V}_1) \bullet (\mathbf{r}_{1,2} - \Delta t' \mathbf{V}_1). \tag{3.2-42}$$

Expanding the right-hand side of (3.2-42) and taking the square root of both sides of the resulting equation yields

$$\left|\mathbf{r}'_{1,2}\right| = r_{1,2}\left[1 - \frac{2}{r_{1,2}}(\hat{n}_{1,2} \bullet \mathbf{V}_1)\Delta t' + \left(\frac{V_1 \Delta t'}{r_{1,2}}\right)^2\right]^{1/2}, \tag{3.2-43}$$

or

$$\left|\mathbf{r}'_{1,2}\right| = r_{1,2}\left\{1 - 2\frac{V_1 \Delta t'}{r_{1,2}}\left[(\hat{n}_{1,2} \bullet \hat{n}_{V_1}) - \frac{1}{2}\frac{V_1 \Delta t'}{r_{1,2}}\right]\right\}^{1/2}. \tag{3.2-44}$$

Equations (3.2-43) and (3.2-44) are *exact* expressions for the range $\left|\mathbf{r}'_{1,2}\right|$. What we need to do next is to derive a valid binomial expansion of the square-root factor in (3.2-44) in order to simplify the expression for the range $\left|\mathbf{r}'_{1,2}\right|$. Equation (3.2-44) can be rewritten as

$$\left|\mathbf{r}'_{1,2}\right| = r_{1,2}\sqrt{1-b} \approx r_{1,2}\left(1 - \frac{b}{2} - \frac{b^2}{8} - \cdots\right), \qquad |b| < 1, \tag{3.2-45}$$

where

$$b = 2\frac{V_1 \Delta t'}{r_{1,2}}\left[(\hat{n}_{1,2} \bullet \hat{n}_{V_1}) - \frac{1}{2}\frac{V_1 \Delta t'}{r_{1,2}}\right]. \tag{3.2-46}$$

Note that (3.2-45) is valid only if $|b| < 1$, where b is given by (3.2-46). If we impose the more stringent criterion that

$$2\frac{V_1 \Delta t'}{r_{1,2}} \le 0.1, \tag{3.2-47}$$

or

$$\boxed{\frac{V_1 \Delta t'}{r_{1,2}} \le 0.05,} \tag{3.2 - 48}$$

then we can use only the first two terms in the binomial expansion. For example, since the maximum negative value of the dot product term in (3.2-46) is -1, and if (3.2-48) is satisfied, then

$$\begin{aligned} |b| &\le 0.1(1 + 0.025) \\ &\le 0.1025 << 1. \end{aligned} \tag{3.2-49}$$

Therefore, if (3.2-48) is satisfied, then we can use only the first two terms in the binomial expansion in (3.2-45). Doing so yields

$$\left|\mathbf{r}'_{1,2}\right| \approx r_{1,2} - V_1 \Delta t'\left[(\hat{n}_{1,2} \bullet \hat{n}_{V_1}) - \frac{1}{2}\frac{V_1 \Delta t'}{r_{1,2}}\right]. \tag{3.2-50}$$

Equation (3.2-50) can be simplified further if we can justify neglecting the second term inside the square brackets. Consider the following criterion: the second term inside the square brackets in (3.2-50) can be neglected if

$$\left|\hat{n}_{1,2} \bullet \hat{n}_{V_1}\right| \geq 10 \frac{1}{2} \frac{V_1 \Delta t'}{r_{1,2}}, \tag{3.2-51}$$

or

$$\boxed{\left|\cos\beta\right| \geq 5 \frac{V_1 \Delta t'}{r_{1,2}},} \tag{3.2-52}$$

where β is the angle between the two unit vectors $\hat{n}_{1,2}$ and $\hat{n}_{V_1}$. The following two ranges of values for the angle β will satisfy (3.2-52):

$$\boxed{0° \leq \beta \leq \cos^{-1}\left(5 \frac{V_1 \Delta t'}{r_{1,2}}\right)} \tag{3.2-53}$$

and

$$\boxed{180° - \cos^{-1}\left(5 \frac{V_1 \Delta t'}{r_{1,2}}\right) \leq \beta \leq 180°.} \tag{3.2-54}$$

For example, if $V_1 \Delta t' / r_{1,2} = 0.05$ [see (3.2-48)], then $0° \leq \beta \leq 75.5°$ and $104.5° \leq \beta \leq 180°$ will satisfy (3.2-52). Therefore, if (3.2-52) is satisfied, then (3.2-50) reduces to

$$\boxed{\left|\mathbf{r}'_{1,2}\right| \approx r_{1,2} - (\hat{n}_{1,2} \bullet \mathbf{V}_1)\Delta t'.} \tag{3.2-55}$$

Equation (3.2-55) is our desired simplified expression for the range $\left|\mathbf{r}'_{1,2}\right|$. We are now in a position to simplify the complex frequency response given by (3.2-23).

We begin the simplification of the complex frequency response given by (3.2-23) by approximating the amplitude factor $1 \big/ \left(\left|\mathbf{r}'_{0,1}\right| \left|\mathbf{r}'_{1,2}\right|\right)$. With the use of (3.2-40) and (3.2-55), we can express the ranges $\left|\mathbf{r}'_{0,1}\right|$ and $\left|\mathbf{r}'_{1,2}\right|$ as follows:

$$\left|\mathbf{r}'_{0,1}\right| \approx r_{0,1}\left[1 + \frac{V_1 \Delta t'}{r_{0,1}} \cos\alpha\right] \tag{3.2-56}$$

and

$$\left|\mathbf{r}'_{1,2}\right| \approx r_{1,2}\left[1 - \frac{V_1 \Delta t'}{r_{1,2}} \cos\beta\right], \tag{3.2-57}$$

where α is the angle between the two unit vectors $\hat{n}_{0,1}$ and $\hat{n}_{V_1}$, and β is the angle between the two unit vectors $\hat{n}_{1,2}$ and $\hat{n}_{V_1}$. With the use of (3.2-33) and (3.2-48), (3.2-56) and (3.2-57) reduce to

$$\left|\mathbf{r}'_{0,1}\right| \approx r_{0,1} \tag{3.2-58}$$

and

$$\left|\mathbf{r}'_{1,2}\right| \approx r_{1,2}, \tag{3.2-59}$$

respectively. Therefore,

$$\boxed{\frac{1}{\left|\mathbf{r}'_{0,1}\right|\left|\mathbf{r}'_{1,2}\right|} \approx \frac{1}{r_{0,1}\, r_{1,2}}.} \tag{3.2-60}$$

Although these approximations of the ranges $\left|\mathbf{r}'_{0,1}\right|$ and $\left|\mathbf{r}'_{1,2}\right|$ are acceptable to approximate the amplitude factor, they are *not* acceptable to approximate the phase factor $\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right]$ in (3.2-23).

Let us approximate the phase factor $\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right]$ next. Adding (3.2-40) and (3.2-55) yields

$$\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right| \approx r_{0,1}+r_{1,2}+\left[(\hat{n}_{0,1}-\hat{n}_{1,2})\bullet \mathbf{V}_1\right]\Delta t', \tag{3.2-61}$$

where $\Delta t'$ is given by (3.2-6). Substituting (3.2-55) into (3.2-25), and since $\Delta t' = t'$ [see (3.2-6)], we obtain

$$\boxed{\Delta t' \approx \frac{1}{1-\beta_{1,2}^{(1)}}\left[t-\frac{r_{1,2}}{c}\right],} \tag{3.2-62}$$

where

$$\boxed{\beta_{1,2}^{(1)} = \frac{\hat{n}_{1,2}\bullet \mathbf{V}_1}{c}} \tag{3.2-63}$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_1$ in the direction $\hat{n}_{1,2}$. Substituting (3.2-62) into (3.2-61) yields

$$\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right| \approx r_{0,1}+r_{1,2}-(s-1)c\left[t-\frac{r_{1,2}}{c}\right], \tag{3.2-64}$$

or

$$\boxed{\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}'_{1,2}\right| \approx r_{0,1} + r_{1,2} - (s-1)ct + (s-1)r_{1,2},} \tag{3.2-65}$$

where

$$\boxed{s = \frac{1-\beta_{0,1}^{(1)}}{1-\beta_{1,2}^{(1)}}} \tag{3.2-66}$$

is the *dimensionless, time-compression / time-stretch factor*, and

$$\boxed{\beta_{0,1}^{(1)} = \frac{\hat{n}_{0,1} \bullet \mathbf{V}_1}{c}} \tag{3.2-67}$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_1$ in the direction $\hat{n}_{0,1}$. With the use of (3.2-65), the phase factor can be expressed as follows:

$$\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}'_{1,2}\right|\right)\right] \approx \exp[+j2\pi f(s-1)t]\exp\left\{-j2\pi f\left[\tau_0 + (s-1)\frac{r_{1,2}}{c}\right]\right\}, \tag{3.2-68}$$

or

$$\boxed{\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}'_{1,2}\right|\right)\right] \approx \exp[+j2\pi fs(t-\tau)]\exp(-j2\pi ft),} \tag{3.2-69}$$

where

$$\boxed{\tau = \frac{1}{s}\left[\tau_0 + (s-1)\frac{r_{1,2}}{c}\right]} \tag{3.2-70}$$

is the *time delay* in seconds (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver), and

$$\boxed{\tau_0 = \frac{r_{0,1} + r_{1,2}}{c}} \tag{3.2-71}$$

is the *time delay* in seconds when *neither* the discrete point scatterer *nor* the transmitter and receiver are in motion.

As we discussed in Section 3.1, in order to model the effects of frequency-dependent attenuation, simply replace the real wavenumber k given by (2.1-23) with the following *complex wavenumber* K [see (3.1-15)]:

$$\boxed{K = k - j\alpha(f),} \tag{3.2-72}$$

where $\alpha(f)$ is the *real, frequency-dependent, attenuation coefficient* in nepers per meter. In addition to being real quantities, both k and $\alpha(f)$ are *positive*. Replacing the real wavenumber k in the phase factor $\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right]$ with the complex wavenumber K given by (3.2-72) yields

$$\exp\left[-jK\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right] = \exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right]\exp\left[-\alpha(f)\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right], \tag{3.2-73}$$

where the phase factor $\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right]$ is given by (3.2-69). As far as the decaying exponential $\exp\left[-\alpha(f)\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right]$ is concerned, we use the following approximation for the distance $\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|$ rather than (3.2-65):

$$\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right| \approx r_{0,1}+r_{1,2} \tag{3.2-74}$$

so that

$$\boxed{\exp\left[-\alpha(f)\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}'_{1,2}\right|\right)\right] \approx \exp\left[-\alpha(f)(r_{0,1}+r_{1,2})\right].} \tag{3.2-75}$$

Equation (3.2-74) can be justified as follows. Since for all real-world problems $V_1/c \ll 1$, the result is that $\left|\beta^{(1)}_{0,1}\right| \ll 1$ and $\left|\beta^{(1)}_{1,2}\right| \ll 1$ [see (3.2-67) and (3.2-63), respectively]. Therefore, $s \approx 1$ [see (3.2-66)] and as a result, the third term in (3.2-64) can be neglected resulting in (3.2-74).

With the use of (3.2-22), (3.2-23), (3.2-60), (3.2-73), (3.2-69), and (3.2-75); we can summarize our results as follows: for the bistatic scattering problem shown in Fig. 3.2-1, the time-harmonic velocity potential in squared-meters per second incident upon the receiver at $\mathbf{r}_2 = (x_2, y_2, z_2)$, due to a time-harmonic point source at $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a moving discrete point scatterer initially at $\mathbf{r}_1 = (x_1, y_1, z_1)$, is given by

$$\boxed{y_M(t,\mathbf{r}_2) = S_0 H_M\left(t,\mathbf{r}_2 \,|\, f,\mathbf{r}_0\right)\exp(+j2\pi ft), \qquad t \geq \tau,} \tag{3.2-76}$$

where S_0 is the *source strength* in cubic meters per second,

$$\boxed{H_M\left(t,\mathbf{r}_2 \,|\, f,\mathbf{r}_0\right) \approx g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)\frac{\exp\left[-\alpha(f)(r_{0,1}+r_{1,2})\right]}{16\pi^2 r_{0,1}\, r_{1,2}}\exp[+j2\pi fs(t-\tau)]\exp(-j2\pi ft)} \tag{3.2-77}$$

is the *time-variant, space-variant, complex frequency response* of the ocean at frequency f hertz, $g_1(f, \hat{n}'_{0,1}, \hat{n}'_{1,2})$ is the *scattering amplitude function* of the discrete point scatterer in meters, $\hat{n}'_{0,1}$ and $\hat{n}'_{1,2}$ are the dimensionless unit vectors in the directions measured from the source to the discrete point scatterer, and from the discrete point scatterer to the receiver, when the transmitted acoustic field is first incident upon the discrete point scatterer at time t', respectively, $\alpha(f)$ is the *real, frequency-dependent, attenuation coefficient* in nepers per meter,

$$\boxed{s = \frac{1-\beta^{(1)}_{0,1}}{1-\beta^{(1)}_{1,2}}} \tag{3.2-78}$$

is the *dimensionless, time-compression / time-stretch factor* [see (3.2-66)],

$$\boxed{\beta^{(1)}_{0,1} = \frac{\hat{n}_{0,1} \bullet \mathbf{V}_1}{c}} \tag{3.2-79}$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_1$ in the direction $\hat{n}_{0,1}$ [see (3.2-67)],

$$\boxed{\beta^{(1)}_{1,2} = \frac{\hat{n}_{1,2} \bullet \mathbf{V}_1}{c}} \tag{3.2-80}$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_1$ in the direction $\hat{n}_{1,2}$ [see (3.2-63)],

$$\boxed{\tau = \frac{1}{s}\left[\tau_0 + (s-1)\frac{r_{1,2}}{c}\right]} \tag{3.2-81}$$

is the *time delay* in seconds (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver) [see (3.2-70)],

$$\boxed{\tau_0 = \frac{r_{0,1} + r_{1,2}}{c}} \tag{3.2-82}$$

is the *time delay* in seconds when *neither* the discrete point scatterer *nor* the transmitter and receiver are in motion [see (3.2-71)], and

$$r_{0,1} = |\mathbf{r}_{0,1}| = |\mathbf{r}_1 - \mathbf{r}_0| \tag{3.2-83}$$

and

$$r_{1,2} = |\mathbf{r}_{1,2}| = |\mathbf{r}_2 - \mathbf{r}_1| \tag{3.2-84}$$

are the distances (ranges) in meters measured from the source to the discrete point scatterer, and from the discrete point scatterer to the receiver, respectively [see (3.1-21) and (3.1-22)], when motion begins at time $t = t_0 = 0$ seconds.

In order to evaluate the scattering amplitude function $g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)$, we need to derive expressions for the unit vectors $\hat{n}'_{0,1}$ and $\hat{n}'_{1,2}$. In general,

$$\hat{n}'_{0,1} = \frac{\mathbf{r}'_{0,1}}{\left|\mathbf{r}'_{0,1}\right|} \tag{3.2-85}$$

and

$$\hat{n}'_{1,2} = \frac{\mathbf{r}'_{1,2}}{\left|\mathbf{r}'_{1,2}\right|}, \tag{3.2-86}$$

where [see (3.2-10)]

$$\mathbf{r}'_{0,1} = r'_{0,1}\hat{n}'_{0,1} = \mathbf{r}_{0,1} + \Delta t'\mathbf{V}_1 \tag{3.2-87}$$

and [see (3.2-18)]

$$\mathbf{r}'_{1,2} = r'_{1,2}\hat{n}'_{1,2} = \mathbf{r}_{1,2} - \Delta t'\mathbf{V}_1, \tag{3.2-88}$$

where

$$r'_{0,1} = \left|\mathbf{r}'_{0,1}\right| \tag{3.2-89}$$

and

$$r'_{1,2} = \left|\mathbf{r}'_{1,2}\right|. \tag{3.2-90}$$

Recall that the parameter $\Delta t'$ is given by [see (3.2-6)]

$$\begin{aligned} \Delta t' &= t' - t_0, \qquad & t' > t_0, \\ &= t', & t' > 0, \end{aligned} \tag{3.2-91}$$

where t' is the time instant when the transmitted acoustic field is first incident upon the discrete point scatterer and $t_0 = 0$ seconds is the time instant when motion begins. What we need to do next is to derive an expression for $\Delta t'$ to be substituted into (3.2-87) and (3.2-88) so that the unit vectors given by (3.2-85) and (3.2-86) can be computed.

The time delay τ (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver) can be expressed as (see Fig. 3.2-2)

$$\tau = \frac{r'_{0,1}}{c} + \frac{r'_{1,2}}{c}. \tag{3.2-92}$$

Since

$$r'_{0,1} = c\Delta t', \tag{3.2-93}$$

substituting (3.2-55) and (3.2-93) into (3.2-92) yields

$$\boxed{\Delta t' \approx \frac{1}{1-\beta_{1,2}^{(1)}}\left[\tau - \frac{r_{1,2}}{c}\right],} \tag{3.2-94}$$

where τ is given by (3.2-81), $r_{1,2}$ is given by (3.2-84), and $\beta_{1,2}^{(1)}$ is given by (3.2-80). Equation (3.2-94) is the desired expression for $\Delta t'$ that will allow us to compute the unit vectors according to (3.2-85) through (3.2-90). Note that (3.2-94) can also be obtained directly from (3.2-62) as follows. If time t in (3.2-62) is set equal to the time instant when the scattered acoustic field *begins* to appear at the receiver, that is, if $t = \tau$, then (3.2-62) reduces to (3.2-94). The parameter $\Delta t'$ given by (3.2-94) is a *constant* whereas $\Delta t'$ given by (3.2-62) is a function of time t where $t \geq \tau$.

The general relationship between the *acoustic pressure* $p(t,\mathbf{r})$ in pascals (Pa) and the velocity potential $y_M(t,\mathbf{r})$ in squared-meters per second is given by

$$\boxed{p(t,\mathbf{r}) = -\rho_0(\mathbf{r})\frac{\partial}{\partial t}y_M(t,\mathbf{r}),} \tag{3.2-95}$$

where $\rho_0(\mathbf{r})$ is the ambient (equilibrium) density of the ocean in kilograms per cubic meter, shown as a function of position $\mathbf{r} = (x,y,z)$. Note that $1\,\text{Pa} = 1\,\text{N/m}^2$ where $1\,\text{N} = 1\,\text{kg-m/sec}^2$. Substituting (3.2-77) into (3.2-76) yields

$$y_M(t,\mathbf{r}_2) \approx S_0\, g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)\frac{\exp\left[-\alpha(f)(r_{0,1}+r_{1,2})\right]}{16\pi^2 r_{0,1}\, r_{1,2}}\exp[+j2\pi fs(t-\tau)], \qquad t \geq \tau. \tag{3.2-96}$$

Substituting (3.2-96) into (3.2-95), and since the unit vectors $\hat{n}'_{0,1}$ and $\hat{n}'_{1,2}$ are constants, and the ambient density is constant in our problem, the acoustic pressure in pascals incident upon the receiver at $\mathbf{r}_2 = (x_2,y_2,z_2)$ is given by

$$\boxed{p(t,\mathbf{r}_2) = -j2\pi fs\rho_0 S_0 H_M\left(t,\mathbf{r}_2\,|\,f,\mathbf{r}_0\right)\exp(+j2\pi ft), \qquad t \geq \tau,} \tag{3.2-97}$$

where $H_M\left(t,\mathbf{r}_2\,|\,f,\mathbf{r}_0\right)$ is given by (3.2-77).

Let us next relate the scattering amplitude function, the differential scattering cross section, and target strength of the discrete point scatterer. The *target strength* (TS) is defined as follows [13]:

$$\boxed{\mathrm{TS} \triangleq 10\log_{10}\left[\frac{\sigma_d\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)}{A_{\mathrm{ref}}}\right]\mathrm{dB\ re\ }A_{\mathrm{ref}},} \tag{3.2-98}$$

where [12, 13]

$$\boxed{\sigma_d\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right) \triangleq \lim_{r'_{1,2}\to\infty}\left[\frac{(r'_{1,2})^2 I_{avg_s}(\mathbf{r}_2)}{I_{avg_i}(\mathbf{r}'_1)}\right] = \frac{\left|g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)\right|^2}{(4\pi)^2}} \tag{3.2-99}$$

is the *differential scattering cross section* with units of squared meters, $I_{avg_i}(\mathbf{r}'_1)$ and $I_{avg_s}(\mathbf{r}_2)$ are the *time-average*, *incident* and *scattered intensities*, respectively, with units of watts per squared meter, $g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)$ is the *scattering amplitude function* of the discrete point scatterer with units of meters, and A_{ref} is a *reference cross-sectional area* commonly chosen to be equal to $1\ \mathrm{m}^2$.

Example 3.2-1 Discrete Point Scatterer In Motion - Monostatic Scattering Geometry

In this example the discrete point scatterer is in motion with constant velocity vector $\mathbf{V}_1$, but the scattering geometry is monostatic versus bistatic. For a *monostatic* (*backscatter*) scattering geometry, both the transmitter and receiver are located at the same position, that is,

$$\mathbf{r}_2 = \mathbf{r}_0. \tag{3.2-100}$$

Therefore (see Fig. 3.2-2),

$$r_{1,2} = r_{0,1}, \tag{3.2-101}$$

$$\hat{n}_{1,2} = -\hat{n}_{0,1}, \tag{3.2-102}$$

$$r'_{1,2} = r'_{0,1}, \tag{3.2-103}$$

and

$$\hat{n}'_{1,2} = -\hat{n}'_{0,1}. \tag{3.2-104}$$

With the use of (3.2-101), (3.2-102), and (3.2-104), the time-variant, space-variant, complex frequency response of the ocean given by (3.2-77) reduces to

$$H_M\left(t,\mathbf{r}_2\,|\,f,\mathbf{r}_0\right) \approx g_1\left(f,\hat{n}'_{0,1},-\hat{n}'_{0,1}\right)\frac{\exp\left[-2\alpha(f)r_{0,1}\right]}{(4\pi r_{0,1})^2}\exp[+j2\pi fs(t-\tau)]\exp(-j2\pi ft), \tag{3.2-105}$$

where $\hat{n}'_{0,1}$ is given by (3.2-85), (3.2-87), and (3.2-94),

$$s = \frac{1 - \beta_{0,1}^{(1)}}{1 + \beta_{0,1}^{(1)}} \tag{3.2-106}$$

is the dimensionless, time-compression / time-stretch factor,

$$\beta_{0,1}^{(1)} = \frac{\hat{n}_{0,1} \bullet \mathbf{V}_1}{c} \tag{3.2-107}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_1$ in the direction $\hat{n}_{0,1}$,

$$\tau = \frac{1}{s}\left[\tau_0 + (s-1)\frac{r_{0,1}}{c}\right] \tag{3.2-108}$$

is the time delay in seconds,

$$\tau_0 = \frac{2r_{0,1}}{c} \tag{3.2-109}$$

is the time delay in seconds when *neither* the discrete point scatterer *nor* the transmitter and receiver are in motion, and $r_{0,1}$ is given by (3.2-83).

Example 3.2-2 No Motion - Bistatic And Monostatic Scattering Geometries

We begin this example by considering the case when the discrete point scatterer is *not* in motion and the scattering geometry is *bistatic*. Therefore,

$$\mathbf{V}_1 = \mathbf{0}, \tag{3.2-110}$$

and as a result,

$$s = 1. \tag{3.2-111}$$

Furthermore (see Fig. 3.2-2),

$$r'_{0,1} = r_{0,1}, \tag{3.2-112}$$

$$\hat{n}'_{0,1} = \hat{n}_{0,1}, \tag{3.2-113}$$

$$r'_{1,2} = r_{1,2}, \tag{3.2-114}$$

and

$$\hat{n}'_{1,2} = \hat{n}_{1,2}. \tag{3.2-115}$$

With the use of (3.2-111), (3.2-113), and (3.2-115), the *time-variant*, space-variant, complex frequency response of the ocean given by (3.2-77) reduces to the exact *time-invariant*, space-variant, complex frequency response of the ocean given by (3.1-17), that is,

$$H_M\left(t,\mathbf{r}_2\,|\,f,\mathbf{r}_0\right)=H_M\left(f,\mathbf{r}_2\,|\,\mathbf{r}_0\right)=g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right)\frac{\exp\left[-\alpha(f)(r_{0,1}+r_{1,2})\right]}{16\pi^2 r_{0,1}r_{1,2}}\exp(-j2\pi f\tau),$$

(3.2-116)

where $\hat{n}_{0,1}$ and $\hat{n}_{1,2}$ are given by (3.1-18) and (3.1-19), respectively,

$$\tau=\frac{r_{0,1}+r_{1,2}}{c} \tag{3.2-117}$$

is the time delay in seconds, and $r_{0,1}$ and $r_{1,2}$ are given by (3.2-83) and (3.2-84), respectively.

If the discrete point scatterer is *not* in motion and the scattering geometry is *monostatic*, then with the use of (3.2-101) and (3.2-102), the time-invariant, space-variant, complex frequency response of the ocean given by (3.2-116) reduces to the exact time-invariant, space-variant, complex frequency response of the ocean given by (3.1-32), that is,

$$H_M\left(t,\mathbf{r}_2\,|\,f,\mathbf{r}_0\right)=H_M\left(f,\mathbf{r}_2\,|\,\mathbf{r}_0\right)=g_1\left(f,\hat{n}_{0,1},-\hat{n}_{0,1}\right)\frac{\exp\left[-2\alpha(f)r_{0,1}\right]}{(4\pi r_{0,1})^2}\exp(-j2\pi f\tau),$$

(3.2-118)

where $\hat{n}_{0,1}$ is given by (3.1-18),

$$\tau=\frac{2r_{0,1}}{c} \tag{3.2-119}$$

is the time delay in seconds, and $r_{0,1}$ is given by (3.2-83).

Example 3.2-3 Pulse Propagation

In this example we will demonstrate how to use the pulse-propagation coupling equations given by (2.2-44) through (2.2-47) for the bistatic scattering problem shown in Fig. 3.2-1. We begin by evaluating (2.2-47), which is repeated below for convenience:

$$\tilde{y}_M(t,\mathbf{r}_{R_n})=\sum_{i=1}^{N_T}\int_{-\infty}^{\infty}\tilde{X}(f)c_{T_i}(f+f_c)S_{T_i}(f+f_c)H_M\left(t,\mathbf{r}_{R_n}\middle|f+f_c,\mathbf{r}_{T_i}\right)\exp(+j2\pi ft)\,df,$$

(2.2-47)

where $\tilde{y}_M(t,\mathbf{r}_{R_n})$ is the complex envelope of the acoustic signal (velocity potential) incident upon element n in the receive array at time t and position $\mathbf{r}_{R_n}=(x_{R_n},y_{R_n},z_{R_n})$ with units of squared-meters per second.

The time-variant, space-variant, complex frequency response of the ocean that describes the bistatic scattering problem shown in Fig. 3.2-1 is given by (3.2-77). Therefore

$$H_M\left(t,\mathbf{r}_{R_n}\middle|f+f_c,\mathbf{r}_{T_i}\right)\approx g_1\left(f+f_c,\hat{n}'_{T_i,1},\hat{n}'_{1,R_n}\right)\frac{\exp\left[-\alpha(f+f_c)(r_{T_i,1}+r_{1,R_n})\right]}{16\pi^2 r_{T_i,1}r_{1,R_n}}\exp\left[+j2\pi(f+f_c)s_{T_i,R_n}(t-\tau_{T_i,R_n})\right]$$
$$\times\exp\left[-j2\pi(f+f_c)t\right],$$

(3.2-120)

where

$$s_{T_i,R_n}=\frac{1-\beta^{(1)}_{T_i,1}}{1-\beta^{(1)}_{1,R_n}} \tag{3.2-121}$$

is the dimensionless, time-compression / time-stretch factor [see (3.2-78)],

$$\beta^{(1)}_{T_i,1}=\frac{\hat{n}_{T_i,1}\bullet\mathbf{V}_1}{c} \tag{3.2-122}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_1$ in the direction $\hat{n}_{T_i,1}$ [see (3.2-79) and (3.1-37)],

$$\beta^{(1)}_{1,R_n}=\frac{\hat{n}_{1,R_n}\bullet\mathbf{V}_1}{c} \tag{3.2-123}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_1$ in the direction $\hat{n}_{1,R_n}$ [see (3.2-80) and (3.1-38)],

$$\tau_{T_i,R_n}=\frac{1}{s_{T_i,R_n}}\left[\tau_0+(s_{T_i,R_n}-1)\frac{r_{1,R_n}}{c}\right] \tag{3.2-124}$$

is the time delay in seconds (the amount of time it takes for the acoustic signal transmitted from element i in the transmit array to *begin* to appear at element n in the receive array) [see (3.2-81)],

$$\tau_0=\frac{r_{T_i,1}+r_{1,R_n}}{c} \tag{3.2-125}$$

is the time delay in seconds when *neither* the discrete point scatterer *nor* the transmit and receive arrays are in motion [see (3.2-82)], and

$$r_{T_i,1}=\left|\mathbf{r}_{T_i,1}\right|=\left|\mathbf{r}_1-\mathbf{r}_{T_i}\right| \tag{3.2-126}$$

and

$$r_{1,R_n}=\left|\mathbf{r}_{1,R_n}\right|=\left|\mathbf{r}_{R_n}-\mathbf{r}_1\right| \tag{3.2-127}$$

are the distances (ranges) in meters measured from element i in the transmit array to the discrete point scatterer, and from the discrete point scatterer to element n in the receive array, respectively [see (3.2-83) and (3.2-84)], when motion begins at time $t = t_0 = 0$ seconds.

In addition, in order to evaluate the scattering amplitude function, we need the following equations:

$$\hat{n}'_{T_i,1} = \frac{\mathbf{r}'_{T_i,1}}{\left|\mathbf{r}'_{T_i,1}\right|} \tag{3.2-128}$$

and

$$\hat{n}'_{1,R_n} = \frac{\mathbf{r}'_{1,R_n}}{\left|\mathbf{r}'_{1,R_n}\right|} \tag{3.2-129}$$

are the dimensionless unit vectors in the directions measured from element i in the transmit array to the discrete point scatterer, and from the discrete point scatterer to element n in the receive array, when the transmitted acoustic field is first incident upon the discrete point scatterer at time t', respectively [see (3.2-85) and (3.2-86)], where [see (3.2-87)]

$$\mathbf{r}'_{T_i,1} = \mathbf{r}_{T_i,1} + \Delta t' \mathbf{V}_1 \tag{3.2-130}$$

and [see (3.2-88)]

$$\mathbf{r}'_{1,R_n} = \mathbf{r}_{1,R_n} - \Delta t' \mathbf{V}_1, \tag{3.2-131}$$

where $\mathbf{r}_{T_i,1} = \mathbf{r}_1 - \mathbf{r}_{T_i}$ and $\mathbf{r}_{1,R_n} = \mathbf{r}_{R_n} - \mathbf{r}_1$ are the position vectors measured from element i in the transmit array to the discrete point scatterer, and from the discrete point scatterer to element n in the receive array, respectively [see (3.1-4) and (3.1-10)], when motion begins at time $t = t_0 = 0$ seconds, where [see (3.2-94)]

$$\Delta t' \approx \frac{1}{1 - \beta^{(1)}_{1,R_n}} \left[\tau_{T_i,R_n} - \frac{r_{1,R_n}}{c} \right]. \tag{3.2-132}$$

Substituting (3.2-120) into (2.2-47) yields

$$\tilde{y}_M(t, \mathbf{r}_{R_n}) \approx \sum_{i=1}^{N_T} \int_{-\infty}^{\infty} \tilde{X}'_{T_i,R_n}(f) \exp\left[+j2\pi f s_{T_i,R_n}(t - \tau_{T_i,R_n})\right] df \exp\left[+j2\pi f_c s_{T_i,R_n}(t - \tau_{T_i,R_n})\right] \exp(-j2\pi f_c t), \tag{3.2-133}$$

or

$$\tilde{y}_M(t, \mathbf{r}_{R_n}) \approx \sum_{i=1}^{N_T} \tilde{x}'_{T_i,R_n}(s_{T_i,R_n}[t - \tau_{T_i,R_n}]) \exp\left[+j2\pi f_c (s_{T_i,R_n} - 1) t\right] \exp(-j2\pi f_c s_{T_i,R_n} \tau_{T_i,R_n}), \tag{3.2-134}$$

where

$$\tilde{X}'_{T_i,R_n}(f) = \tilde{X}(f)c_{T_i}(f+f_c)S_{T_i}(f+f_c)g_1\left(f+f_c,\hat{n}'_{T_i,1},\hat{n}'_{1,R_n}\right)\frac{\exp\left[-\alpha(f+f_c)(r_{T_i,1}+r_{1,R_n})\right]}{16\pi^2 r_{T_i,1}r_{1,R_n}} \tag{3.2-135}$$

and

$$\tilde{x}'_{T_i,R_n}(t) = F_f^{-1}\left\{\tilde{X}'_{T_i,R_n}(f)\right\} = \int_{-\infty}^{\infty}\tilde{X}'_{T_i,R_n}(f)\exp(+j2\pi ft)\,df. \tag{3.2-136}$$

Equation (3.2-134) indicates that the complex envelope, $\tilde{y}_M(t,\mathbf{r}_{R_n})$, of the acoustic signal (velocity potential) incident upon element n in the receive array is equal to the sum of time-compressed / time-stretched, time-delayed, Doppler shifted, modified versions of the complex envelopes of the transmitted electrical signals from each element in the transmit array. The amount of Doppler shift in hertz is given by

$$\phi_{T_i,R_n} = (s_{T_i,R_n} - 1)f_c. \tag{3.2-137}$$

Equation (3.2-135) shows how the frequency spectrum of the complex envelope of the transmitted electrical signal, $\tilde{X}(f)$, gets modified by various other important frequency-dependent functions in order to produce the complex envelope, $\tilde{y}_M(t,\mathbf{r}_{R_n})$, of the acoustic signal (velocity potential) incident upon element n in the receive array.

Before continuing with the analysis, several comments concerning the dimensionless, time-compression / time-stretch factor s_{T_i,R_n} and (3.2-137) are in order. There are *three* ranges of values for s_{T_i,R_n}; namely, $s_{T_i,R_n}=1$, $s_{T_i,R_n}>1$, and $s_{T_i,R_n}<1$. When $s_{T_i,R_n}=1$, this is an indication that there is *no* time compression or time stretch, and as a result, there is *no* change in signal bandwidth. Also, when $s_{T_i,R_n}=1$, $\phi_{T_i,R_n}=0$; that is, there is *no* Doppler shift [see (3.2-137)]. For example, if there is *no* motion, then the velocity vector of the discrete point scatterer is equal to the zero vector. Therefore, both beta parameters are equal to zero and, as a result, $s_{T_i,R_n}=1$. When $s_{T_i,R_n}>1$, this is an indication of *time compression*, that is, the duration of the received signal given by (3.2-134) is *decreased* (compared to the duration of the transmitted signal) resulting in an *increase* in signal bandwidth. Also, when $s_{T_i,R_n}>1$, $\phi_{T_i,R_n}>0$; which indicates a *positive* Doppler shift [see (3.2-137)]. Finally, when $s_{T_i,R_n}<1$, this is an indication of *time stretch*, that is, the duration of the received signal is *increased* (compared to the duration of the transmitted signal) resulting in a *decrease* in signal bandwidth. Also, when $s_{T_i,R_n}<1$, $\phi_{T_i,R_n}<0$; which indicates a *negative* Doppler shift [see (3.2-137)].

If we next substitute (3.2-134) into (2.2-46), then we obtain

$$y_M(t,\mathbf{r}_{R_n}) \approx \sum_{i=1}^{N_T}\operatorname{Re}\left\{\tilde{x}'_{T_i,R_n}(s_{T_i,R_n}[t-\tau_{T_i,R_n}])\exp\left[+j2\pi f_c s_{T_i,R_n}(t-\tau_{T_i,R_n})\right]\right\}. \tag{3.2-138}$$

Since $x(t)$ given by (2.2-28) and $\tilde{x}(t)$ given by (2.2-30) are related by

$$x(t) = \operatorname{Re}\left\{\tilde{x}(t)\exp(+j2\pi f_c t)\right\}, \tag{3.2-139}$$

if we let

$$x'_{T_i,R_n}(t) = \mathrm{Re}\left\{\tilde{x}'_{T_i,R_n}(t)\exp(+j2\pi f_c t)\right\}, \tag{3.2-140}$$

then (3.2-138) can be rewritten as

$$y_M(t,\mathbf{r}_{R_n}) \approx \sum_{i=1}^{N_T} x'_{T_i,R_n}(s_{T_i,R_n}[t - \tau_{T_i,R_n}]). \tag{3.2-141}$$

Substituting (3.2-141) into (2.2-45) yields

$$Y_M(\eta,\mathbf{r}_{R_n}) \approx \sum_{i=1}^{N_T} \frac{1}{\left|s_{T_i,R_n}\right|} X'_{T_i,R_n}(\eta/s_{T_i,R_n})\exp(-j2\pi\eta\tau_{T_i,R_n}), \tag{3.2-142}$$

and upon substituting (3.2-142) into (2.2-44), we obtain

$$y(t,\mathbf{r}_{R_n}) \approx \sum_{i=1}^{N_T} \int_{-\infty}^{\infty} X''_{T_i,R_n}(\eta)\exp\left[+j2\pi\eta(t - \tau_{T_i,R_n})\right]d\eta, \qquad n = 1,2,\ldots,N_R, \tag{3.2-143}$$

or, finally,

$$y(t,\mathbf{r}_{R_n}) \approx \sum_{i=1}^{N_T} x''_{T_i,R_n}(t - \tau_{T_i,R_n}), \qquad n = 1,2,\ldots,N_R, \tag{3.2-144}$$

where

$$X''_{T_i,R_n}(\eta) = \frac{1}{\left|s_{T_i,R_n}\right|} X'_{T_i,R_n}(\eta/s_{T_i,R_n})c_{R_n}(\eta)S_{R_n}(\eta) \tag{3.2-145}$$

and

$$x''_{T_i,R_n}(t) = F_\eta^{-1}\left\{X''_{T_i,R_n}(\eta)\right\} = \int_{-\infty}^{\infty} X''_{T_i,R_n}(\eta)\exp(+j2\pi\eta t)\,d\eta. \tag{3.2-146}$$

Equation (3.2-144) indicates that the output electrical signal $y(t,\mathbf{r}_{R_n})$ from element n in the receive array at time t and position $\mathbf{r}_{R_n} = (x_{R_n}, y_{R_n}, z_{R_n})$ with units of volts is equal to the sum of time-delayed, modified versions of the transmitted electrical signals from each element in the transmit array.

3.3 All Three Platforms In Motion

In this section we will analyze the bistatic scattering problem shown in Fig. 3.3-1. All three platforms - the transmitter, discrete point scatterer, and receiver - are in motion. For example purposes, they are placed in *deep water* so that the ocean medium can be treated as being *unbounded*, that is, there are *no* boundaries and, hence, *no* reflections. The speed of sound and

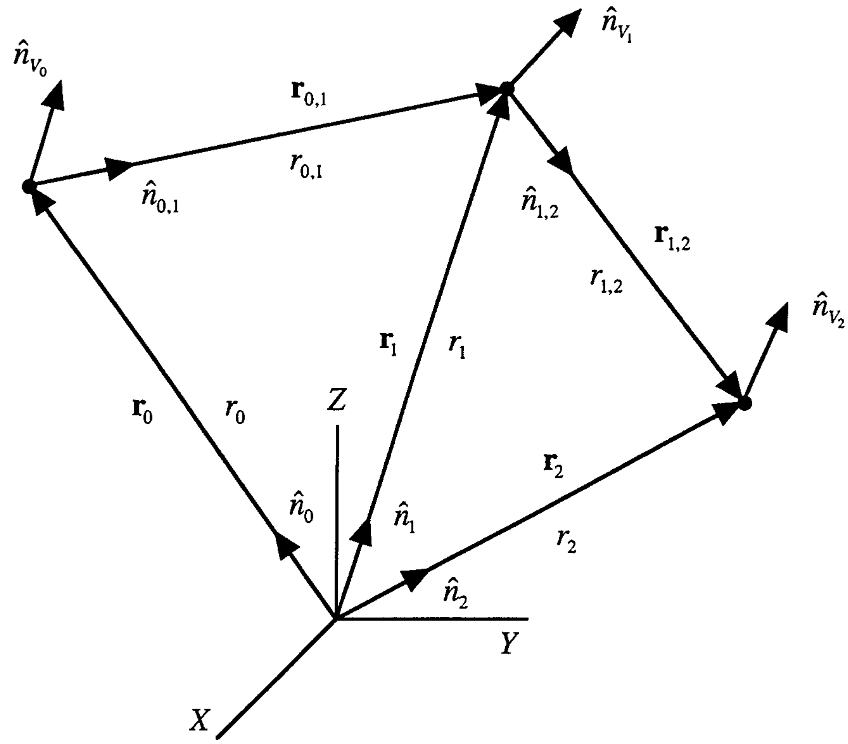

Figure 3.3-1. Bistatic scattering geometry when motion begins at time $t = t_0 = 0$ seconds. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. All three platforms are in motion.

ambient density are *constants.* Although the propagation of sound between the source and discrete point scatterer, and between the discrete point scatterer and receiver can be treated as transmission through linear, time-variant, *space-invariant* filters; the overall solution for this bistatic scattering problem corresponds to transmission through a linear, time-variant, *space-variant* filter. The presence of a discrete point scatterer in an unbounded, homogeneous fluid medium (i.e., a fluid medium with constant speed of sound and ambient density) causes the medium to be space-variant.

The *velocity vectors* of the transmitter, $\mathbf{V}_0$, the discrete point scatterer, $\mathbf{V}_1$, and the receiver, $\mathbf{V}_2$, are given by

$$\mathbf{V}_0 = V_0 \hat{n}_{V_0}, \tag{3.3-1}$$

$$\mathbf{V}_1 = V_1 \hat{n}_{V_1}, \tag{3.3-2}$$

and

$$\mathbf{V}_2 = V_2 \hat{n}_{V_2}, \tag{3.3-3}$$

where V_0, V_1, and V_2 are the *speeds* in meters per second of the transmitter, the discrete point scatterer, and the receiver, respectively, and $\hat{n}_{V_0}$, $\hat{n}_{V_1}$, and $\hat{n}_{V_2}$ are the dimensionless unit vectors in the directions of $\mathbf{V}_0$, $\mathbf{V}_1$, and $\mathbf{V}_2$, respectively. The velocity vectors given by (3.3-1) through (3.3-3) are *constant*, that is, the speeds and directions are *constants* - there is *no* acceleration. Motion begins at time $t = t_0 = 0$ seconds.

Since all three platforms are now in motion, the position vectors from the origin to the transmitter, discrete point scatterer, and receiver - denoted by $\mathcal{R}_0(t)$, $\mathcal{R}_1(t)$, and $\mathcal{R}_2(t)$, respectively - are functions of time given by

$$\mathcal{R}_0(t) = \mathbf{r}_0 + \Delta t\,\mathbf{V}_0, \qquad t \geq 0, \tag{3.3-4}$$

$$\mathcal{R}_1(t) = \mathbf{r}_1 + \Delta t\,\mathbf{V}_1, \qquad t \geq 0, \tag{3.3-5}$$

and

$$\mathcal{R}_2(t) = \mathbf{r}_2 + \Delta t\,\mathbf{V}_2, \qquad t \geq 0, \tag{3.3-6}$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$, $\mathbf{r}_1 = (x_1, y_1, z_1)$, and $\mathbf{r}_2 = (x_2, y_2, z_2)$ are the position vectors from the origin to the transmitter, discrete point scatterer, and receiver, respectively, when motion begins at time $t = t_0 = 0$ seconds (see Fig. 3.3-1), and

$$\begin{aligned} \Delta t &= t - t_0, \qquad && t \geq t_0, \\ &= t, && t \geq 0. \end{aligned} \tag{3.3-7}$$

Note that $\mathcal{R}_0(0) = \mathbf{r}_0$, $\mathcal{R}_1(0) = \mathbf{r}_1$, and $\mathcal{R}_2(0) = \mathbf{r}_2$. Instead of trying to solve this bistatic scattering problem directly with all three platforms in motion, we will first create an equivalent problem involving the transmitter and the discrete point scatterer where we can treat the transmitter (sound source) as being motionless. We will then create a second equivalent problem involving the discrete point scatterer and the receiver where we can treat the discrete point scatterer (acting as a sound source) as being motionless. This can be accomplished by working with *relative velocity vectors.*

When motion begins at time $t = t_0 = 0$ seconds, the *scalar* component of $\mathbf{V}_0$ in the direction of $\mathbf{V}_1$ is given by (see Fig. 3.3-2)

$$\hat{n}_{V_1} \bullet \mathbf{V}_0 = \hat{n}_{V_1} \bullet V_0 \hat{n}_{V_0} = V_0(\hat{n}_{V_1} \bullet \hat{n}_{V_0}). \tag{3.3-8}$$

Therefore, the velocity vector of the discrete point scatterer *relative* to the velocity vector of the transmitter *in the direction of the velocity vector of the discrete point scatterer* $\hat{n}_{V_1}$ is given by

$$\boxed{\mathbf{V}_{1,0} = \mathbf{V}_1 - (\hat{n}_{V_1} \bullet \mathbf{V}_0)\hat{n}_{V_1} = \left[V_1 - V_0(\hat{n}_{V_1} \bullet \hat{n}_{V_0})\right]\hat{n}_{V_1}.} \tag{3.3-9}$$

By using the *relative velocity vector* $\mathbf{V}_{1,0}$, the transmitter (sound source) can be treated as being *motionless*. Therefore, when motion begins at time $t = t_0 = 0$ seconds, the source distribution $x_M(t,\mathbf{r})$ will be treated as a *motionless, time-harmonic, point source* with units of inverse seconds, that is, let [see (2.1-8) and (2.1-11)]

$$x_M(t,\mathbf{r}) = S_0\delta(\mathbf{r} - \mathbf{r}_0)\exp(+j2\pi ft), \tag{3.3-10}$$

where S_0 is the *source strength* in cubic meters per second and the impulse function $\delta(\mathbf{r} - \mathbf{r}_0)$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_0 = (x_0, y_0, z_0)$. In addition, since we are now working with the relative velocity vector $\mathbf{V}_{1,0}$, we need to introduce the new position vector

$$\mathcal{R}_1^{(1,0)}(t) = \mathbf{r}_1 + \Delta t\,\mathbf{V}_{1,0}, \qquad t \geq 0, \tag{3.3-11}$$

where Δt is given by (3.3-7). Compare (3.3-11) with (3.3-5). Note that $\mathcal{R}_1^{(1,0)}(0) = \mathbf{r}_1$. Also note that if the transmitter is *not* in motion, then $\mathbf{V}_0 = \mathbf{0}$, and as a result, $\mathbf{V}_{1,0} = \mathbf{V}_1$ [see (3.3-9)] and $\mathcal{R}_1^{(1,0)}(t) = \mathcal{R}_1(t)$ [see (3.3-11) and (3.3-5)]. And if the discrete point scatterer is *not* in motion, then $\mathbf{V}_1 = \mathbf{0}$ and the relative velocity vector $\mathbf{V}_{1,0}$ given by (3.3-9) is *undefined.* In this case we set $\mathbf{V}_{1,0} = \mathbf{0}$, and as a result, $\mathcal{R}_1^{(1,0)}(t) = \mathcal{R}_1(t) = \mathbf{r}_1$.

When the transmitted acoustic field is first incident upon the discrete point scatterer at some time t' seconds where $t' > 0$, the position vector from the origin to the discrete point scatterer is given by [see (3.3-11) and Fig. 3.3-3]

$$\mathbf{r}_1' = \mathcal{R}_1^{(1,0)}(t') = \mathbf{r}_1 + \Delta t'\,\mathbf{V}_{1,0}, \qquad t' > 0, \tag{3.3-12}$$

where

$$\begin{aligned} \Delta t' &= t' - t_0, \qquad & t' > t_0, \\ &= t', \qquad & t' > 0. \end{aligned} \tag{3.3-13}$$

The propagation of sound between the source and discrete point scatterer can be modeled as transmission through a linear, *time-variant*, *space-invariant* filter. Therefore, the acoustic field (velocity potential) incident upon the discrete point scatterer at time t' and position $\mathbf{r}_1' = (x_1', y_1', z_1')$ is given by [see (2.1-62)]

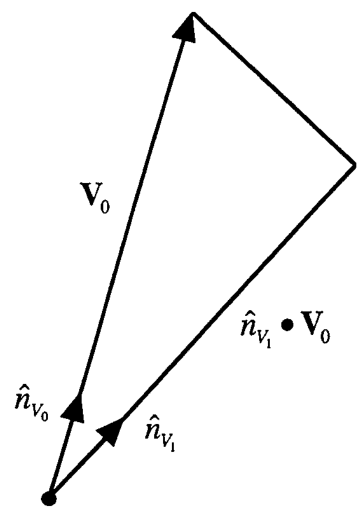

Figure 3.3-2. Scalar component of $\mathbf{V}_0$ in the direction of $\mathbf{V}_1$.

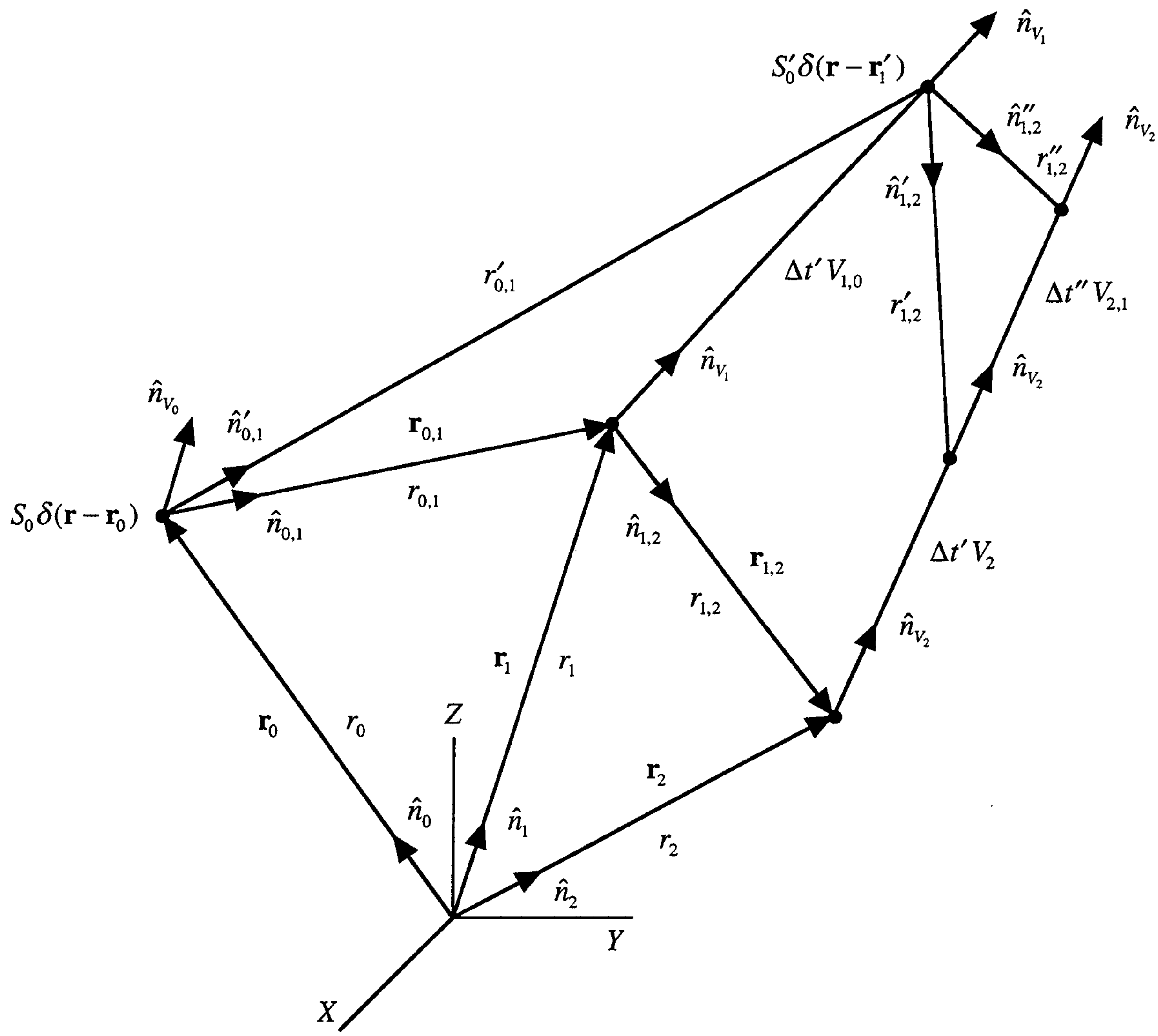

Figure 3.3-3. Bistatic scattering geometry when the transmitted acoustic field is first incident upon the discrete point scatterer at time t' seconds and when the scattered acoustic field is first incident upon the receiver at time t seconds where $t > t' > t_0$ $(t_0 = 0)$. Point 0, $P_0(\mathbf{r}_0)$, is the transmitter; point 1, $P_1(\mathbf{r}_1)$, is the discrete point scatterer; and point 2, $P_2(\mathbf{r}_2)$, is the receiver. All three platforms are in motion.

$$
\begin{aligned}
y_M(t',\mathbf{r}_1') &= S_0 H_M\left(t',\mathbf{r}_1'-\mathbf{r}_0 \,|\, f\right)\exp(+j2\pi f t') \\
&= -S_0 \frac{\exp\left(-jk\left|\mathbf{r}_1'-\mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}_1'-\mathbf{r}_0\right|}\exp(+j2\pi f t'), \qquad t'>0,
\end{aligned}
\tag{3.3-14}
$$

where

$$
H_M\left(t',\mathbf{r}_1'-\mathbf{r}_0 \,|\, f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_1'-\mathbf{r}_0\right|\right)}{4\pi\left|\mathbf{r}_1'-\mathbf{r}_0\right|}. \tag{3.3-15}
$$

By referring to Fig. 3.3-3, we can express the position vector from the point source to the discrete point scatterer at time t' as

$$
\mathbf{r}_{0,1}' = \mathbf{r}_1' - \mathbf{r}_0, \tag{3.3-16}
$$

and upon substituting (3.3-12) into (3.3-16), we obtain

$$
\mathbf{r}_{0,1}' = r_{0,1}'\hat{n}_{0,1}' = \mathbf{r}_{0,1} + \Delta t'\mathbf{V}_{1,0}, \tag{3.3-17}
$$

where $\mathbf{r}_{0,1}$ is given by (3.1-4). Therefore, (3.3-14) and (3.3-15) can be rewritten as

$$
\begin{aligned}
y_M(t',\mathbf{r}_1') &= S_0 H_M\left(t',\mathbf{r}_{0,1}' \,|\, f\right)\exp(+j2\pi f t') \\
&= -S_0 \frac{\exp\left(-jk\left|\mathbf{r}_{0,1}'\right|\right)}{4\pi\left|\mathbf{r}_{0,1}'\right|}\exp(+j2\pi f t'), \qquad t'>0,
\end{aligned}
\tag{3.3-18}
$$

and

$$
H_M\left(t',\mathbf{r}_{0,1}' \,|\, f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_{0,1}'\right|\right)}{4\pi\left|\mathbf{r}_{0,1}'\right|}. \tag{3.3-19}
$$

Let us now create a similar equivalent problem involving the discrete point scatterer and the receiver where we can treat the discrete point scatterer (acting as a sound source) as being motionless. When the transmitted acoustic field is first incident upon the discrete point scatterer at time $t' > 0$ seconds, the position vector from the origin to the receiver is given by [see (3.3-6) and Fig. 3.3-3]

$$
\mathbf{r}_2' = \mathcal{R}_2(t') = \mathbf{r}_2 + \Delta t'\mathbf{V}_2, \qquad t'>0, \tag{3.3-20}
$$

where $\Delta t'$ is given by (3.3-13). Equation (3.3-20) indicates that after $\Delta t'$ seconds, the receiver - independent of the transmitter and the discrete point scatterer - travels an additional distance of $\Delta t' V_2$ meters in the direction $\hat{n}_{V_2}$ (see Fig. 3.3-3). Also at time t', the *scalar* component of $\mathbf{V}_1$ in the direction of $\mathbf{V}_2$ is given by (see Fig. 3.3-4)

$$
\hat{n}_{V_2} \bullet \mathbf{V}_1 = \hat{n}_{V_2} \bullet V_1\hat{n}_{V_1} = V_1(\hat{n}_{V_2} \bullet \hat{n}_{V_1}). \tag{3.3-21}
$$

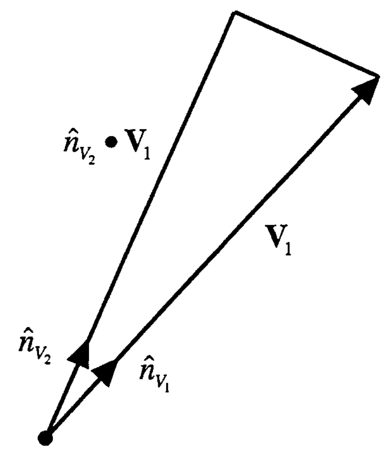

Figure 3.3-4. Scalar component of $\mathbf{V}_1$ in the direction of $\mathbf{V}_2$.

Therefore, the velocity vector of the receiver *relative* to the velocity vector of the discrete point scatterer *in the direction of the velocity vector of the receiver* $\hat{n}_{V_2}$ is given by

$$\boxed{\mathbf{V}_{2,1} = \mathbf{V}_2 - (\hat{n}_{V_2} \bullet \mathbf{V}_1)\hat{n}_{V_2} = \left[V_2 - V_1(\hat{n}_{V_2} \bullet \hat{n}_{V_1})\right]\hat{n}_{V_2}.} \tag{3.3-22}$$

By using the *relative velocity vector* $\mathbf{V}_{2,1}$, the discrete point scatterer (acting as a sound source) can be treated as being motionless. Therefore, in order to compute the acoustic signal at the receiver, we treat the discrete point scatterer at time $t \geq t'$ and position $\mathbf{r}_1'$ as another *motionless, time-harmonic, point source* with units of inverse seconds, that is, let [see (3.3-10) and Fig. 3.3-3]

$$x_M'(t,\mathbf{r}) = S_0'\delta(\mathbf{r} - \mathbf{r}_1')\exp(+j2\pi ft), \tag{3.3-23}$$

where S_0' is the *source strength* in cubic meters per second and the impulse function $\delta(\mathbf{r} - \mathbf{r}_1')$, with units of inverse cubic meters, represents a point source at $\mathbf{r}_1' = (x_1', y_1', z_1')$, where $\mathbf{r}_1'$ is given by (3.3-12). The source strength S_0' will be given later.

Since we are now working with the relative velocity vector $\mathbf{V}_{2,1}$, we need to introduce the new position vector

$$\mathcal{R}_2^{(2,1)}(t) = \mathbf{r}_2' + \Delta t''\mathbf{V}_{2,1}, \qquad t \geq t' > 0, \tag{3.3-24}$$

where

$$\Delta t'' = t - t', \qquad t \geq t' > 0, \tag{3.3-25}$$

and $\mathbf{r}_2'$ is given by (3.3-20). Compare (3.3-24) with (3.3-6). Note that $\mathcal{R}_2^{(2,1)}(t') = \mathbf{r}_2'$. Also note that if the discrete point scatterer is *not* in motion, then $\mathbf{V}_1 = \mathbf{0}$, and as a result, $\mathbf{V}_{2,1} = \mathbf{V}_2$ [see (3.3-22)] and $\mathcal{R}_2^{(2,1)}(t) = \mathcal{R}_2(t)$ for $t \geq t' > 0$ [see (3.3-24) and (3.3-6)]. And if the receiver is *not* in motion, then $\mathbf{V}_2 = \mathbf{0}$ and the relative velocity vector $\mathbf{V}_{2,1}$ given by (3.3-22) is *undefined.* In this case we set $\mathbf{V}_{2,1} = \mathbf{0}$, and as a result, $\mathcal{R}_2^{(2,1)}(t) = \mathcal{R}_2(t) = \mathbf{r}_2$ for $t \geq t' > 0$.

When the scattered acoustic field is first incident upon the receiver at some time t seconds where $t > t' > 0$, the position vector from the origin to the receiver is given by [see (3.3-24) and Fig. 3.3-3]

$$\mathbf{r}_2'' = \mathcal{R}_2^{(2,1)}(t) = \mathbf{r}_2' + \Delta t''\mathbf{V}_{2,1}, \qquad t > t' > 0, \tag{3.3-26}$$

where $\Delta t''$ is given by (3.3-25). Equation (3.3-26) indicates that after $\Delta t''$ seconds, the receiver travels an additional distance of $\Delta t'' V_{2,1}$ meters in the direction $\hat{n}_{V_2}$ (see Fig. 3.3-3). By referring to Fig. 3.3-3, it can be seen that

$$\mathbf{r}_{1,2}'' = \mathbf{r}_2'' - \mathbf{r}_1', \tag{3.3-27}$$

where $\mathbf{r}_{1,2}''$ is the position vector between the discrete point scatterer and the receiver at time $t > t' > 0$. Substituting (3.3-12) and (3.3-26) into (3.3-27) yields

$$\mathbf{r}_{1,2}'' = r_{1,2}''\hat{n}_{1,2}'' = \mathbf{r}_{1,2} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t''\mathbf{V}_{2,1}, \tag{3.3-28}$$

where $\mathbf{r}_{1,2}$ is given by (3.1-10). Equation (3.3-28) can also be obtained directly from Fig. 3.3-3 - without the use of (3.3-27) - simply by using vector addition. Doing so yields

$$\begin{aligned} \mathbf{r}_{1,2}'' = r_{1,2}''\hat{n}_{1,2}'' &= -\Delta t'\mathbf{V}_{1,0} + \mathbf{r}_{1,2} + \Delta t'\mathbf{V}_2 + \Delta t''\mathbf{V}_{2,1} \\ &= \mathbf{r}_{1,2} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t''\mathbf{V}_{2,1}. \end{aligned} \tag{3.3-29}$$

The propagation of sound between the discrete point scatterer and receiver can also be modeled as transmission through a linear, *time-variant, space-invariant* filter. Therefore, the acoustic field (velocity potential) incident upon the receiver at time t and position $\mathbf{r}_2'' = (x_2'', y_2'', z_2'')$ due to a point source at $\mathbf{r}_1' = (x_1', y_1', z_1')$ is given by [see (2.1-62)]

$$\begin{aligned} y_M(t, \mathbf{r}_2'') &= S_0' H_M\left(t, \mathbf{r}_2'' - \mathbf{r}_1' \,\middle|\, f\right)\exp(+j2\pi f t) \\ &= -S_0' \frac{\exp\left(-jk\left|\mathbf{r}_2'' - \mathbf{r}_1'\right|\right)}{4\pi\left|\mathbf{r}_2'' - \mathbf{r}_1'\right|}\exp(+j2\pi f t), \qquad t > t' > 0, \end{aligned} \tag{3.3-30}$$

where

$$H_M\left(t, \mathbf{r}_2'' - \mathbf{r}_1' \,\middle|\, f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_2'' - \mathbf{r}_1'\right|\right)}{4\pi\left|\mathbf{r}_2'' - \mathbf{r}_1'\right|}. \tag{3.3-31}$$

With the use of (3.3-27), (3.3-30) and (3.3-31) can be rewritten as

$$\begin{aligned} y_M(t, \mathbf{r}_2'') &= S_0' H_M\left(t, \mathbf{r}_{1,2}'' \,\middle|\, f\right)\exp(+j2\pi f t) \\ &= -S_0' \frac{\exp\left(-jk\left|\mathbf{r}_{1,2}''\right|\right)}{4\pi\left|\mathbf{r}_{1,2}''\right|}\exp(+j2\pi f t), \qquad t > t' > 0, \end{aligned} \tag{3.3-32}$$

and

$$H_M\left(t, \mathbf{r}_{1,2}'' \,\middle|\, f\right) = -\frac{\exp\left(-jk\left|\mathbf{r}_{1,2}''\right|\right)}{4\pi\left|\mathbf{r}_{1,2}''\right|}. \tag{3.3-33}$$

The source strength S_0' is given by [see (3.3-18)]

$$S_0' = S_0 H_M\left(t', \mathbf{r}_{0,1}' \,\middle|\, f\right) g_1\left(f, \hat{n}_{0,1}', \hat{n}_{1,2}''\right), \tag{3.3-34}$$

where $g_1\left(f, \hat{n}_{0,1}', \hat{n}_{1,2}''\right)$ is the *scattering amplitude function* of the discrete point scatterer with units of meters as discussed in Section 3.1. The unit of meters for the scattering amplitude function represents an *effective scattering length* that may be larger or smaller than the actual length of the scatterer. As we discussed in Section 3.1, the scattering amplitude function is a complex function (magnitude and phase) and is, in general, a function of frequency, the direction of wave propagation from the source to the scatterer, and the direction of wave propagation from the scatterer to the

receiver [12]. Since the transmitted acoustic field is first incident upon the discrete point scatterer at time t', the direction of wave propagation from the source to the scatterer is given by the dimensionless unit vector $\hat{n}'_{0,1}$ (see Fig. 3.3-3). Similarly, since the discrete point scatterer is being treated as another point source at time t' and position $\mathbf{r}'_1$, and the scattered acoustic field is first incident upon the receiver at time $t > t' > 0$, the direction of wave propagation from the scatterer to the receiver is given by the dimensionless unit vector $\hat{n}''_{1,2}$ (see Fig. 3.3-3).

Let us now begin the process of obtaining final expressions for both the time-harmonic velocity potential and acoustic pressure incident upon the receiver. Substituting (3.3-34) into (3.3-32) yields

$$y_M(t,\mathbf{r}''_2) = S_0 H_M\left(t',\mathbf{r}'_{0,1}\middle|f\right) g_1\left(f,\hat{n}'_{0,1},\hat{n}''_{1,2}\right) H_M\left(t,\mathbf{r}''_{1,2}\middle|f\right)\exp(+j2\pi ft), \qquad t > t' > 0, \tag{3.3-35}$$

or, equivalently,

$$y_M(t,\mathbf{r}''_2) = S_0 H_M\left(t,\mathbf{r}''_2\,|\,f,\mathbf{r}_0\right)\exp(+j2\pi ft), \qquad t > t' > 0, \tag{3.3-36}$$

where

$$\begin{aligned} H_M\left(t,\mathbf{r}''_2\,|\,f,\mathbf{r}_0\right) &= H_M\left(t',\mathbf{r}'_{0,1}\middle|f\right) g_1\left(f,\hat{n}'_{0,1},\hat{n}''_{1,2}\right) H_M\left(t,\mathbf{r}''_{1,2}\middle|f\right) \\ &= g_1\left(f,\hat{n}'_{0,1},\hat{n}''_{1,2}\right)\frac{\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right]}{16\pi^2\left|\mathbf{r}'_{0,1}\right|\left|\mathbf{r}''_{1,2}\right|} \\ &= g_1\left(f,\hat{n}'_{0,1},\hat{n}''_{1,2}\right)\frac{\exp\left[-jk\left(\left|\mathbf{r}_1 - \mathbf{r}_0 + \Delta t'\mathbf{V}_{1,0}\right| + \left|\mathbf{r}_2 - \mathbf{r}_1 + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t''\mathbf{V}_{2,1}\right|\right)\right]}{16\pi^2\left|\mathbf{r}_1 - \mathbf{r}_0 + \Delta t'\mathbf{V}_{1,0}\right|\left|\mathbf{r}_2 - \mathbf{r}_1 + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t''\mathbf{V}_{2,1}\right|}, \end{aligned} \tag{3.3-37}$$

$\Delta t'$ is given by (3.3-13), $\mathbf{V}_{1,0}$ is given by (3.3-9), $\Delta t''$ is given by (3.3-25), $\mathbf{V}_{2,1}$ is given by (3.3-22), and

$$t = t' + \frac{\left|\mathbf{r}''_{1,2}\right|}{c}, \tag{3.3-38}$$

or

$$\boxed{t' = t - \frac{\left|\mathbf{r}''_{1,2}\right|}{c}.} \tag{3.3-39}$$

Note that if the bistatic scattering problem shown in Fig. 3.3-1 corresponded to transmission through a space-invariant filter, then the complex frequency response given by (3.3-37) would be a function of the vector spatial difference $\mathbf{r}''_2 - \mathbf{r}_0$, which it is *not* [$\mathbf{r}''_2$ is given by (3.3-26)].

In order to simplify the complex frequency response given by (3.3-37), we need to derive approximate, simplified expressions for the ranges $\left|\mathbf{r}'_{0,1}\right|$ and $\left|\mathbf{r}''_{1,2}\right|$. Let us begin with $\left|\mathbf{r}'_{0,1}\right|$. Since

$$\left|\mathbf{r}'_{0,1}\right|^2 = \mathbf{r}'_{0,1} \bullet \mathbf{r}'_{0,1}, \tag{3.3-40}$$

substituting (3.3-17) into (3.3-40) yields

$$\left|\mathbf{r}'_{0,1}\right|^2 = (\mathbf{r}_{0,1} + \Delta t' \mathbf{V}_{1,0}) \bullet (\mathbf{r}_{0,1} + \Delta t' \mathbf{V}_{1,0}). \tag{3.3-41}$$

Expanding the right-hand side of (3.3-41) and taking the square root of both sides of the resulting equation yields

$$\left|\mathbf{r}'_{0,1}\right| = r_{0,1}\left[1 + \frac{2}{r_{0,1}}(\hat{n}_{0,1} \bullet \mathbf{V}_{1,0})\Delta t' + \left(\frac{V_{1,0}\Delta t'}{r_{0,1}}\right)^2\right]^{1/2}, \tag{3.3-42}$$

or

$$\left|\mathbf{r}'_{0,1}\right| = r_{0,1}\left\{1 + 2\frac{V_{1,0}\Delta t'}{r_{0,1}}\left[(\hat{n}_{0,1} \bullet \hat{n}_{V_1}) + \frac{1}{2}\frac{V_{1,0}\Delta t'}{r_{0,1}}\right]\right\}^{1/2}. \tag{3.3-43}$$

Equations (3.3-42) and (3.3-43) are *exact* expressions for the range $\left|\mathbf{r}'_{0,1}\right|$. What we need to do next is to derive a valid binomial expansion of the square-root factor in (3.3-43) in order to simplify the expression for the range $\left|\mathbf{r}'_{0,1}\right|$. Equation (3.3-43) can be rewritten as

$$\left|\mathbf{r}'_{0,1}\right| = r_{0,1}\sqrt{1+b} \approx r_{0,1}\left(1 + \frac{b}{2} - \frac{b^2}{8} + \cdots\right), \qquad |b| < 1, \tag{3.3-44}$$

where

$$b = 2\frac{V_{1,0}\Delta t'}{r_{0,1}}\left[(\hat{n}_{0,1} \bullet \hat{n}_{V_1}) + \frac{1}{2}\frac{V_{1,0}\Delta t'}{r_{0,1}}\right]. \tag{3.3-45}$$

Note that (3.3-44) is valid only if $|b| < 1$, where b is given by (3.3-45). If we impose the more stringent criterion that

$$2\frac{V_{1,0}\Delta t'}{r_{0,1}} \le 0.1, \tag{3.3-46}$$

or

$$\boxed{\frac{V_{1,0}\Delta t'}{r_{0,1}} \le 0.05,} \tag{3.3-47}$$

then we can use only the first two terms in the binomial expansion. For example, since the maximum positive value of the dot product term in (3.3-45) is $+1$, and if (3.3-47) is satisfied, then

$$\begin{aligned} |b| &\le 0.1(1+0.025) \\ &\le 0.1025 \ll 1. \end{aligned} \tag{3.3-48}$$

Therefore, if (3.3-47) is satisfied, then we can use only the first two terms in the binomial expansion in (3.3-44). Doing so yields

$$\left|\mathbf{r}'_{0,1}\right| \approx r_{0,1} + V_{1,0}\Delta t'\left[(\hat{n}_{0,1} \bullet \hat{n}_{V_1}) + \frac{1}{2}\frac{V_{1,0}\Delta t'}{r_{0,1}}\right]. \tag{3.3-49}$$

Equation (3.3-49) can be simplified further if we can justify neglecting the second term inside the square brackets. Consider the following criterion: the second term inside the square brackets in (3.3-49) can be neglected if

$$\left|\hat{n}_{0,1} \bullet \hat{n}_{V_1}\right| \ge 10\frac{1}{2}\frac{V_{1,0}\Delta t'}{r_{0,1}}, \tag{3.3-50}$$

or

$$\boxed{\left|\cos\alpha\right| \ge 5\frac{V_{1,0}\Delta t'}{r_{0,1}},} \tag{3.3-51}$$

where α is the angle between the two unit vectors $\hat{n}_{0,1}$ and $\hat{n}_{V_1}$. The following two ranges of values for the angle α will satisfy (3.3-51):

$$\boxed{0° \le \alpha \le \cos^{-1}\left(5\frac{V_{1,0}\Delta t'}{r_{0,1}}\right)} \tag{3.3-52}$$

and

$$\boxed{180° - \cos^{-1}\left(5\frac{V_{1,0}\Delta t'}{r_{0,1}}\right) \le \alpha \le 180°.} \tag{3.3-53}$$

For example, if $V_{1,0}\Delta t'/r_{0,1} = 0.05$ [see (3.3-47)], then $0° \le \alpha \le 75.5°$ and $104.5° \le \alpha \le 180°$ will satisfy (3.3-51). Therefore, if (3.3-51) is satisfied, then (3.3-49) reduces to

$$\boxed{\left|\mathbf{r}'_{0,1}\right| \approx r_{0,1} + (\hat{n}_{0,1} \bullet \mathbf{V}_{1,0})\Delta t'.} \tag{3.3-54}$$

Equation (3.3-54) is our desired simplified expression for the range $\left|\mathbf{r}'_{0,1}\right|$. Let us next derive a similar expression for the range $\left|\mathbf{r}''_{1,2}\right|$.

Since

$$\left|\mathbf{r}''_{1,2}\right|^2 = \mathbf{r}''_{1,2} \bullet \mathbf{r}''_{1,2}, \tag{3.3-55}$$

substituting (3.3-29) into (3.3-55) yields

$$\left|\mathbf{r}''_{1,2}\right|^2 = \left[\mathbf{r}_{1,2} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t''\mathbf{V}_{2,1}\right] \bullet \left[\mathbf{r}_{1,2} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t''\mathbf{V}_{2,1}\right]. \tag{3.3-56}$$

Expanding the right-hand side of (3.3-56) and taking the square root of both sides of the resulting equation yields

$$\left|\mathbf{r}''_{1,2}\right| = r_{1,2}\left\{1 + \frac{2}{r_{1,2}}\left[\hat{n}_{1,2} \bullet (\mathbf{V}_2 - \mathbf{V}_{1,0})\right]\Delta t' + \left(\frac{\left|\mathbf{V}_2 - \mathbf{V}_{1,0}\right|\Delta t'}{r_{1,2}}\right)^2 + \frac{2}{r_{1,2}}(\hat{n}_{1,2} \bullet \mathbf{V}_{2,1})\Delta t'' + \left(\frac{V_{2,1}\Delta t''}{r_{1,2}}\right)^2 \right.$$
$$\left. + 2\frac{\Delta t'\Delta t''}{r_{1,2}^2}\left[\mathbf{V}_{2,1} \bullet (\mathbf{V}_2 - \mathbf{V}_{1,0})\right]\right\}^{1/2}, \tag{3.3-57}$$

or

$$\left|\mathbf{r}''_{1,2}\right| = r_{1,2}\left\{1 - 2\frac{V_{1,0}\Delta t'}{r_{1,2}}\left[(\hat{n}_{1,2} \bullet \hat{n}_{V_1}) + \frac{\Delta t'}{2r_{1,2}}\left(V_2(\hat{n}_{V_1} \bullet \hat{n}_{V_2}) - V_{1,0}\right)\right]\right.$$
$$+ 2\frac{V_2\Delta t'}{r_{1,2}}\left[(\hat{n}_{1,2} \bullet \hat{n}_{V_2}) + \frac{\Delta t'}{2r_{1,2}}\left(V_2 - V_{1,0}(\hat{n}_{V_1} \bullet \hat{n}_{V_2})\right)\right]$$
$$\left. + 2\frac{V_{2,1}\Delta t''}{r_{1,2}}\left[(\hat{n}_{1,2} \bullet \hat{n}_{V_2}) + \frac{V_{2,1}\Delta t''}{2r_{1,2}} + \frac{\Delta t'}{r_{1,2}}\left(V_2 - V_{1,0}(\hat{n}_{V_1} \bullet \hat{n}_{V_2})\right)\right]\right\}^{1/2}. \tag{3.3-58}$$

Equations (3.3-57) and (3.3-58) are *exact* expressions for the range $\left|\mathbf{r}''_{1,2}\right|$. What we need to do next is to derive a valid binomial expansion of the square-root factor in (3.3-57) in order to simplify the expression for the range $\left|\mathbf{r}''_{1,2}\right|$. We begin the simplification process by neglecting those terms in (3.3-57) involving $\Delta t'$ and $\Delta t''$ squared and the cross product term involving $\Delta t'\Delta t''$. Doing so yields

$$\left|\mathbf{r}_{1,2}''\right| \approx r_{1,2}\sqrt{1+b} \approx r_{1,2}\left(1+\frac{b}{2}-\frac{b^2}{8}+\cdots\right), \qquad |b|<1, \tag{3.3-59}$$

where

$$b = \frac{2}{r_{1,2}}\left\{\left[\hat{n}_{1,2} \bullet (\mathbf{V}_2 - \mathbf{V}_{1,0})\right]\Delta t' + (\hat{n}_{1,2} \bullet \mathbf{V}_{2,1})\Delta t''\right\}. \tag{3.3-60}$$

Note that the binomial expansion in (3.3-59) is valid only if $|b|<1$, where b is given by (3.3-60). Assuming that $|b|<1$ and if we use only the first two terms in the binomial expansion in (3.3-59), then

$$\boxed{\left|\mathbf{r}_{1,2}''\right| \approx r_{1,2} + \left[\hat{n}_{1,2} \bullet (\mathbf{V}_2 - \mathbf{V}_{1,0})\right]\Delta t' + (\hat{n}_{1,2} \bullet \mathbf{V}_{2,1})\Delta t''.} \tag{3.3-61}$$

Equation (3.3-61) is our desired simplified expression for the range $\left|\mathbf{r}_{1,2}''\right|$. We are now in a position to simplify the complex frequency response given by (3.3-37).

We begin the simplification of the complex frequency response given by (3.3-37) by approximating the amplitude factor $1/\left(\left|\mathbf{r}_{0,1}'\right|\left|\mathbf{r}_{1,2}''\right|\right)$. With the use of (3.3-54) and (3.3-61), we can express the ranges $\left|\mathbf{r}_{0,1}'\right|$ and $\left|\mathbf{r}_{1,2}''\right|$ as follows:

$$\left|\mathbf{r}_{0,1}'\right| \approx r_{0,1}\left[1+\frac{V_{1,0}\Delta t'}{r_{0,1}}\cos\alpha\right], \tag{3.3-62}$$

where α is the angle between the two unit vectors $\hat{n}_{0,1}$ and $\hat{n}_{V_1}$, and

$$\left|\mathbf{r}_{1,2}''\right| \approx r_{1,2}\left[1+\frac{\left[\hat{n}_{1,2} \bullet (\mathbf{V}_2 - \mathbf{V}_{1,0})\right]\Delta t'}{r_{1,2}}+\frac{(\hat{n}_{1,2} \bullet \mathbf{V}_{2,1})\Delta t''}{r_{1,2}}\right]. \tag{3.3-63}$$

With the use of (3.3-47) in (3.3-62), and upon neglecting the second and third terms inside the square brackets in (3.3-63), (3.3-62) and (3.3-63) reduce to

$$\left|\mathbf{r}_{0,1}'\right| \approx r_{0,1} \tag{3.3-64}$$

and

$$\left|\mathbf{r}_{1,2}''\right| \approx r_{1,2}, \tag{3.3-65}$$

respectively. Therefore,

$$\boxed{\frac{1}{\left|\mathbf{r}_{0,1}'\right|\left|\mathbf{r}_{1,2}''\right|} \approx \frac{1}{r_{0,1}\, r_{1,2}}.} \tag{3.3-66}$$

Although these approximations of the ranges $\left|\mathbf{r}'_{0,1}\right|$ and $\left|\mathbf{r}''_{1,2}\right|$ are acceptable to approximate the amplitude factor, they are *not* acceptable to approximate the phase factor $\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}''_{1,2}\right|\right)\right]$ in (3.3-37).

Let us approximate the phase factor $\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}''_{1,2}\right|\right)\right]$ next. Adding (3.3-54) and (3.3-61) yields

$$\left|\mathbf{r}'_{0,1}\right|+\left|\mathbf{r}''_{1,2}\right| \approx r_{0,1}+r_{1,2}+\left\{\left[(\hat{n}_{0,1}-\hat{n}_{1,2})\bullet\mathbf{V}_{1,0}\right]+(\hat{n}_{1,2}\bullet\mathbf{V}_{2})\right\}\Delta t'+(\hat{n}_{1,2}\bullet\mathbf{V}_{2,1})\Delta t'', \quad (3.3\text{-}67)$$

where $\Delta t'$ is given by (3.3-13) and $\Delta t''$ is given by (3.3-25). Substituting (3.3-61) into (3.3-39), and since $\Delta t' = t'$ [see (3.3-13)], we obtain

$$\Delta t' \approx t-\frac{r_{1,2}}{c}-\frac{\hat{n}_{1,2}\bullet(\mathbf{V}_2-\mathbf{V}_{1,0})}{c}\Delta t'-\frac{\hat{n}_{1,2}\bullet\mathbf{V}_{2,1}}{c}\Delta t''. \quad (3.3\text{-}68)$$

The parameter $\Delta t''$, given by (3.3-25), can be rewritten as follows:

$$\begin{aligned}\Delta t'' &= t-t' \\ &= t-t_0-(t'-t_0), \qquad t>t'>t_0,\end{aligned} \quad (3.3\text{-}69)$$

or

$$\begin{aligned}\Delta t'' &= t-t_0-\Delta t', \qquad t>t'>t_0, \\ &= t-\Delta t', \qquad t>t'>0,\end{aligned} \quad (3.3\text{-}70)$$

where $\Delta t'$ is given by (3.3-13). Substituting (3.3-70) into (3.3-68) yields

$$\boxed{\Delta t' \approx \frac{1}{1-\beta_{1,2}^{(1,0)}+\beta_{1,2}^{(2)}-\beta_{1,2}^{(2,1)}}\left\{\left[1-\beta_{1,2}^{(2,1)}\right]t-\frac{r_{1,2}}{c}\right\},} \quad (3.3\text{-}71)$$

where

$$\boxed{\beta_{1,2}^{(2)}=\frac{\hat{n}_{1,2}\bullet\mathbf{V}_2}{c}} \quad (3.3\text{-}72)$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_2$ in the direction $\hat{n}_{1,2}$,

$$\boxed{\beta_{1,2}^{(1,0)}=\frac{\hat{n}_{1,2}\bullet\mathbf{V}_{1,0}}{c}} \quad (3.3\text{-}73)$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_{1,0}$ in the direction $\hat{n}_{1,2}$, and

$$\boxed{\beta_{1,2}^{(2,1)} = \frac{\hat{n}_{1,2} \bullet \mathbf{V}_{2,1}}{c}} \tag{3.3-74}$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_{2,1}$ in the direction $\hat{n}_{1,2}$. Substituting (3.3-70) and (3.3-71) into (3.3-67) yields

$$\boxed{\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right| \approx r_{0,1} + r_{1,2} - (s-1)ct + \frac{s - 1 + \beta_{1,2}^{(2,1)}}{1 - \beta_{1,2}^{(2,1)}} r_{1,2},} \tag{3.3-75}$$

where

$$\boxed{s = \frac{\left[1 - \beta_{0,1}^{(1,0)}\right]\left[1 - \beta_{1,2}^{(2,1)}\right]}{\left[1 - \beta_{1,2}^{(1,0)}\right] + \left[\beta_{1,2}^{(2)} - \beta_{1,2}^{(2,1)}\right]}} \tag{3.3-76}$$

is the *dimensionless, time-compression / time-stretch factor*, and

$$\boxed{\beta_{0,1}^{(1,0)} = \frac{\hat{n}_{0,1} \bullet \mathbf{V}_{1,0}}{c}} \tag{3.3-77}$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_{1,0}$ in the direction $\hat{n}_{0,1}$. With the use of (3.3-75), the phase factor can be expressed as follows:

$$\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right] \approx \exp[+j2\pi f(s-1)t]\exp\left\{-j2\pi f\left[\tau_0 + \frac{s - 1 + \beta_{1,2}^{(2,1)}}{1 - \beta_{1,2}^{(2,1)}}\frac{r_{1,2}}{c}\right]\right\}, \tag{3.3-78}$$

or

$$\boxed{\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right] \approx \exp[+j2\pi fs(t-\tau)]\exp(-j2\pi ft),} \tag{3.3-79}$$

where

$$\boxed{\tau = \frac{1}{s}\left[\tau_0 + \frac{s - 1 + \beta_{1,2}^{(2,1)}}{1 - \beta_{1,2}^{(2,1)}}\frac{r_{1,2}}{c}\right]} \tag{3.3-80}$$

is the *time delay* in seconds (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver), and

$$\boxed{\tau_0 = \frac{r_{0,1} + r_{1,2}}{c}} \tag{3.3-81}$$

is the *time delay* in seconds when *neither* the discrete point scatterer *nor* the transmitter and receiver are in motion.

As we discussed in Section 3.1, in order to model the effects of frequency-dependent attenuation, simply replace the real wavenumber k given by (2.1-23) with the following *complex wavenumber* K [see (3.1-15)]:

$$\boxed{K = k - j\alpha(f),} \tag{3.3-82}$$

where $\alpha(f)$ is the *real, frequency-dependent, attenuation coefficient* in nepers per meter. In addition to being real quantities, both k and $\alpha(f)$ are *positive*. Replacing the real wavenumber k in the phase factor $\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right]$ with the complex wavenumber K given by (3.3-82) yields

$$\exp\left[-jK\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right] = \exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right]\exp\left[-\alpha(f)\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right], \tag{3.3-83}$$

where the phase factor $\exp\left[-jk\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right]$ is given by (3.3-79). As far as the decaying exponential $\exp\left[-\alpha(f)\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right]$ is concerned, we use the following approximation for the distance $\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|$ rather than (3.3-75):

$$\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}'_{1,2}\right| \approx r_{0,1} + r_{1,2} \tag{3.3-84}$$

so that

$$\boxed{\exp\left[-\alpha(f)\left(\left|\mathbf{r}'_{0,1}\right| + \left|\mathbf{r}''_{1,2}\right|\right)\right] \approx \exp\left[-\alpha(f)(r_{0,1} + r_{1,2})\right].} \tag{3.3-85}$$

Equation (3.3-84) can be justified as follows. Since for all real-world problems $V_{1,0}/c \ll 1$, $V_{2,1}/c \ll 1$, and $V_2/c \ll 1$, the result is that $\left|\beta_{0,1}^{(1,0)}\right| \ll 1$, $\left|\beta_{1,2}^{(2,1)}\right| \ll 1$, $\left|\beta_{1,2}^{(1,0)}\right| \ll 1$, and $\left|\beta_{1,2}^{(2)}\right| \ll 1$ [see (3.3-77), (3.3-74), (3.3-73), and (3.3-72), respectively]. Therefore, $s \approx 1$ [see (3.3-76)] and since $\left|\beta_{1,2}^{(2,1)}\right| \ll 1$, the third and fourth terms in (3.3-75) can be neglected resulting in (3.3-84).

With the use of (3.3-36), (3.3-37), (3.3-66), (3.3-83), (3.3-79), and (3.3-85); we can summarize our results as follows: for the bistatic scattering problem shown in Fig. 3.3-1, the time-harmonic velocity potential in squared-meters per second incident upon the receiver at

$\mathbf{r}_2'' = (x_2'', y_2'', z_2'')$, due to a moving time-harmonic point source initially at $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a moving discrete point scatterer initially at $\mathbf{r}_1 = (x_1, y_1, z_1)$, is given by

$$\boxed{y_M(t, \mathbf{r}_2'') = S_0 H_M\left(t, \mathbf{r}_2'' \,|\, f, \mathbf{r}_0\right) \exp(+j2\pi f t), \qquad t \geq \tau,} \tag{3.3-86}$$

where S_0 is the *source strength* in cubic meters per second,

$$\boxed{H_M\left(t, \mathbf{r}_2'' \,|\, f, \mathbf{r}_0\right) \approx g_1\left(f, \hat{n}_{0,1}', \hat{n}_{1,2}''\right) \frac{\exp\left[-\alpha(f)(r_{0,1} + r_{1,2})\right]}{16\pi^2 r_{0,1} r_{1,2}} \exp[+j2\pi f s(t-\tau)] \exp(-j2\pi f t)} \tag{3.3-87}$$

is the *time-variant, space-variant, complex frequency response* of the ocean at frequency f hertz, $g_1\left(f, \hat{n}_{0,1}', \hat{n}_{1,2}''\right)$ is the *scattering amplitude function* of the discrete point scatterer in meters, $\hat{n}_{0,1}'$ is the dimensionless unit vector in the direction measured from the source to the discrete point scatterer when the transmitted acoustic field is first incident upon the discrete point scatterer, $\hat{n}_{1,2}''$ is the dimensionless unit vector in the direction measured from the discrete point scatterer to the receiver when the scattered acoustic field is first incident upon the receiver, $\alpha(f)$ is the *real, frequency-dependent, attenuation coefficient* in nepers per meter,

$$\boxed{s = \frac{\left[1 - \beta_{0,1}^{(1,0)}\right]\left[1 - \beta_{1,2}^{(2,1)}\right]}{\left[1 - \beta_{1,2}^{(1,0)}\right] + \left[\beta_{1,2}^{(2)} - \beta_{1,2}^{(2,1)}\right]}} \tag{3.3-88}$$

is the *dimensionless, time-compression / time-stretch factor* [see (3.3-76)],

$$\boxed{\beta_{0,1}^{(1,0)} = \frac{\hat{n}_{0,1} \bullet \mathbf{V}_{1,0}}{c}} \tag{3.3-89}$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_{1,0}$ in the direction $\hat{n}_{0,1}$ [see (3.3-77)],

$$\boxed{\beta_{1,2}^{(2,1)} = \frac{\hat{n}_{1,2} \bullet \mathbf{V}_{2,1}}{c}} \tag{3.3-90}$$

is the *dimensionless, normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_{2,1}$ in the direction $\hat{n}_{1,2}$ [see (3.3-74)],

$$\boxed{\beta_{1,2}^{(1,0)} = \frac{\hat{n}_{1,2} \bullet \mathbf{V}_{1,0}}{c}} \tag{3.3-91}$$

is the *dimensionless*, *normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_{1,0}$ in the direction $\hat{n}_{1,2}$ [see (3.3-73)],

$$\boxed{\beta_{1,2}^{(2)} = \frac{\hat{n}_{1,2} \bullet \mathbf{V}_2}{c}} \tag{3.3-92}$$

is the *dimensionless*, *normalized* (normalized by the constant speed of sound c) *radial-component* of $\mathbf{V}_2$ in the direction $\hat{n}_{1,2}$ [see (3.3-72)],

$$\boxed{\mathbf{V}_{1,0} = \mathbf{V}_1 - (\hat{n}_{V_1} \bullet \mathbf{V}_0)\hat{n}_{V_1} = \left[V_1 - V_0(\hat{n}_{V_1} \bullet \hat{n}_{V_0})\right]\hat{n}_{V_1}} \tag{3.3-93}$$

is the velocity vector of the discrete point scatterer *relative* to the velocity vector of the transmitter *in the direction of the velocity vector of the discrete point scatterer* $\hat{n}_{V_1}$ [see (3.3-9)],

$$\boxed{\mathbf{V}_{2,1} = \mathbf{V}_2 - (\hat{n}_{V_2} \bullet \mathbf{V}_1)\hat{n}_{V_2} = \left[V_2 - V_1(\hat{n}_{V_2} \bullet \hat{n}_{V_1})\right]\hat{n}_{V_2}} \tag{3.3-94}$$

is the velocity vector of the receiver *relative* to the velocity vector of the discrete point scatterer *in the direction of the velocity vector of the receiver* $\hat{n}_{V_2}$ [see (3.3-22)],

$$\boxed{\tau = \frac{1}{s}\left[\tau_0 + \frac{s - 1 + \beta_{1,2}^{(2,1)}}{1 - \beta_{1,2}^{(2,1)}}\frac{r_{1,2}}{c}\right]} \tag{3.3-95}$$

is the *time delay* in seconds (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver) [see (3.3-80)],

$$\boxed{\tau_0 = \frac{r_{0,1} + r_{1,2}}{c}} \tag{3.3-96}$$

is the *time delay* in seconds when *neither* the discrete point scatterer *nor* the transmitter and receiver are in motion [see (3.3-81)], and

$$r_{0,1} = \left|\mathbf{r}_{0,1}\right| = \left|\mathbf{r}_1 - \mathbf{r}_0\right| \tag{3.3-97}$$

and

$$r_{1,2} = \left|\mathbf{r}_{1,2}\right| = \left|\mathbf{r}_2 - \mathbf{r}_1\right| \tag{3.3-98}$$

are the distances (ranges) in meters measured from the source to the discrete point scatterer, and from the discrete point scatterer to the receiver, respectively [see (3.1-21) and (3.1-22)], when motion begins at time $t = t_0 = 0$ seconds. Recall that if the discrete point scatterer is *not* in motion,

then $\mathbf{V}_1 = \mathbf{0}$ and the relative velocity vector $\mathbf{V}_{1,0}$ given by (3.3-93) is *undefined.* In this case we set $\mathbf{V}_{1,0} = \mathbf{0}$. Similarly, if the receiver is *not* in motion, then $\mathbf{V}_2 = \mathbf{0}$ and the relative velocity vector $\mathbf{V}_{2,1}$ given by (3.3-94) is *undefined.* In this case we set $\mathbf{V}_{2,1} = \mathbf{0}$. The position vector from the origin to the receiver when the scattered acoustic field is incident upon the receiver is given by [see (3.3-26), (3.3-20), and (3.3-70)]

$$\boxed{\mathbf{r}_2'' = \mathbf{r}_2 + \Delta t' \mathbf{V}_2 + (t - \Delta t')\mathbf{V}_{2,1}, \qquad t \geq \tau,} \tag{3.3-99}$$

where $\Delta t'$ is given by (3.3-71) and the relative velocity vector $\mathbf{V}_{2,1}$ is given by (3.3-94).

In order to evaluate the scattering amplitude function $g_1\left(f, \hat{n}_{0,1}', \hat{n}_{1,2}''\right)$, we need to derive expressions for the unit vectors $\hat{n}_{0,1}'$ and $\hat{n}_{1,2}''$. In general,

$$\hat{n}_{0,1}' = \frac{\mathbf{r}_{0,1}'}{\left|\mathbf{r}_{0,1}'\right|} \tag{3.3-100}$$

and

$$\hat{n}_{1,2}'' = \frac{\mathbf{r}_{1,2}''}{\left|\mathbf{r}_{1,2}''\right|}, \tag{3.3-101}$$

where [see (3.3-17)]

$$\mathbf{r}_{0,1}' = r_{0,1}' \hat{n}_{0,1}' = \mathbf{r}_{0,1} + \Delta t' \mathbf{V}_{1,0} \tag{3.3-102}$$

and [see (3.3-28)]

$$\mathbf{r}_{1,2}'' = r_{1,2}'' \hat{n}_{1,2}'' = \mathbf{r}_{1,2} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t'' \mathbf{V}_{2,1}, \tag{3.3-103}$$

where

$$r_{0,1}' = \left|\mathbf{r}_{0,1}'\right| \tag{3.3-104}$$

and

$$r_{1,2}'' = \left|\mathbf{r}_{1,2}''\right|. \tag{3.3-105}$$

Recall that the parameters $\Delta t'$ and $\Delta t''$ are given by [see (3.3-13) and (3.3-70)]

$$\begin{aligned} \Delta t' &= t' - t_0, \qquad & t' &> t_0, \\ &= t', & t' &> 0, \end{aligned} \tag{3.3-106}$$

and

$$\Delta t'' = t - t_0 - \Delta t', \qquad t > t' > t_0,$$
$$= t - \Delta t', \qquad t > t' > 0, \tag{3.3-107}$$

where t' is the time instant when the transmitted acoustic field is first incident upon the discrete point scatterer and $t_0 = 0$ is the time instant when motion begins. If time t in (3.3-107) is set equal to the time instant when the scattered acoustic field *begins* to appear at the receiver, that is, if $t = \tau$, then (3.3-107) reduces to

$$\Delta t'' = \tau - \Delta t'. \tag{3.3-108}$$

What we need to do next is to derive an expression for $\Delta t'$ to be substituted into (3.3-102), (3.3-108), and (3.3-103) so that the unit vectors given by (3.3-100) and (3.3-101) can be computed.

The time delay τ (the amount of time it takes for the transmitted acoustic signal to *begin* to appear at the receiver) can be expressed as (see Fig. 3.3-3)

$$\tau = \frac{r'_{0,1}}{c} + \frac{r''_{1,2}}{c}. \tag{3.3-109}$$

Since

$$r'_{0,1} = c\Delta t', \tag{3.3-110}$$

substituting (3.3-61), (3.3-108), and (3.3-110) into (3.3-109) yields

$$\boxed{\Delta t' \approx \frac{1}{1 - \beta_{1,2}^{(1,0)} + \beta_{1,2}^{(2)} - \beta_{1,2}^{(2,1)}} \left\{ \left[1 - \beta_{1,2}^{(2,1)}\right]\tau - \frac{r_{1,2}}{c} \right\},} \tag{3.3-111}$$

where τ is given by (3.3-95), $r_{1,2}$ is given by (3.3-98), and the three beta parameters are given by (3.3-90) through (3.3-92). Equation (3.3-111) is the desired expression for $\Delta t'$ that will allow us to compute the unit vectors according to (3.3-100) through (3.3-105) and (3.3-108). Note that (3.3-111) can also be obtained directly from (3.3-71) as follows. If time t in (3.3-71) is set equal to the time instant when the scattered acoustic field *begins* to appear at the receiver, that is, if $t = \tau$, then (3.3-71) reduces to (3.3-111). The parameter $\Delta t'$ given by (3.3-111) is a *constant* whereas $\Delta t'$ given by (3.3-71) is a function of time t where $t \geq \tau$.

The general relationship between the *acoustic pressure* $p(t,\mathbf{r})$ in pascals (Pa) and the velocity potential $y_M(t,\mathbf{r})$ in squared-meters per second is given by

$$\boxed{p(t,\mathbf{r}) = -\rho_0(\mathbf{r}) \frac{\partial}{\partial t} y_M(t,\mathbf{r}),} \tag{3.3-112}$$

where $\rho_0(\mathbf{r})$ is the ambient (equilibrium) density of the ocean in kilograms per cubic meter, shown as a function of position $\mathbf{r} = (x, y, z)$. Note that $1\,\text{Pa} = 1\,\text{N/m}^2$ where $1\,\text{N} = 1\,\text{kg-m/sec}^2$. Substituting (3.3-87) into (3.3-86) yields

$$y_M(t,\mathbf{r}_2'') \approx S_0 g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right)\frac{\exp\left[-\alpha(f)(r_{0,1}+r_{1,2})\right]}{16\pi^2 r_{0,1} r_{1,2}}\exp[+j2\pi fs(t-\tau)], \qquad t \ge \tau.$$

(3.3-113)

Substituting (3.3-113) into (3.3-112), and since the unit vectors $\hat{n}_{0,1}'$ and $\hat{n}_{1,2}''$ are constants, and the ambient density is constant in our problem, the acoustic pressure in pascals incident upon the receiver at $\mathbf{r}_2'' = (x_2'', y_2'', z_2'')$ is given by

$$\boxed{p(t,\mathbf{r}_2'') = -j2\pi fs\rho_0 S_0 H_M\left(t,\mathbf{r}_2''\left|f,\mathbf{r}_0\right.\right)\exp(+j2\pi ft), \qquad t \ge \tau,} \tag{3.3-114}$$

where $H_M\left(t,\mathbf{r}_2''\left|f,\mathbf{r}_0\right.\right)$ is given by (3.3-87).

Let us next relate the scattering amplitude function, the differential scattering cross section, and target strength of the discrete point scatterer. The *target strength* (TS) is defined as follows [13]:

$$\boxed{\mathrm{TS} \triangleq 10\log_{10}\left[\frac{\sigma_d\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right)}{A_{\mathrm{ref}}}\right]\text{dB re } A_{\mathrm{ref}},} \tag{3.3-115}$$

where [12, 13]

$$\boxed{\sigma_d\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right) \triangleq \lim_{r_{1,2}''\to\infty}\left[\frac{(r_{1,2}'')^2 I_{avg_s}(\mathbf{r}_2'')}{I_{avg_i}(\mathbf{r}_1')}\right] = \frac{\left|g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right)\right|^2}{(4\pi)^2}} \tag{3.3-116}$$

is the *differential scattering cross section* with units of squared meters, $I_{avg_i}(\mathbf{r}_1')$ and $I_{avg_s}(\mathbf{r}_2'')$ are the *time-average, incident* and *scattered intensities*, respectively, with units of watts per squared meter, $g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right)$ is the *scattering amplitude function* of the discrete point scatterer with units of meters, and A_{ref} is a *reference cross-sectional area* commonly chosen to be equal to $1\ \mathrm{m}^2$.

Example 3.3-1 Monostatic Scattering Geometry

We begin this example by considering the case when *all* three platforms are in motion with constant velocity vectors $\mathbf{V}_0$, $\mathbf{V}_1$, and $\mathbf{V}_2$, but the scattering geometry is monostatic versus bistatic. For a *monostatic* (*backscatter*) scattering geometry, both the transmitter and receiver are located at the same position, that is,

$$\mathbf{r}_2 = \mathbf{r}_0. \tag{3.3-117}$$

Therefore (see Fig. 3.3-1),

$$\mathbf{V}_2 = \mathbf{V}_0, \tag{3.3-118}$$

$$r_{1,2} = r_{0,1}, \tag{3.3-119}$$

and

$$\hat{n}_{1,2} = -\hat{n}_{0,1}. \tag{3.3-120}$$

With the use of (3.3-118) through (3.3-120), the time-variant, space-variant, complex frequency response of the ocean given by (3.3-87) reduces to

$$H_M\left(t,\mathbf{r}_2''\,|\,f,\mathbf{r}_0\right) \approx g_1\left(f,\hat{n}_{0,1}',\hat{n}_{1,2}''\right)\frac{\exp\left[-2\alpha(f)r_{0,1}\right]}{(4\pi r_{0,1})^2}\exp[+j2\pi fs(t-\tau)]\exp(-j2\pi ft), \tag{3.3-121}$$

where $\mathbf{r}_2''$ is given by (3.3-99) and (3.3-111), $\hat{n}_{0,1}'$ is given by (3.3-100), (3.3-102), and (3.3-111), $\hat{n}_{1,2}''$ is given by (3.3-101), (3.3-103), (3.3-108), and (3.3-111), $\beta_{1,2}^{(2,1)} = -\beta_{0,1}^{(2,1)}$, $\beta_{1,2}^{(1,0)} = -\beta_{0,1}^{(1,0)}$, $\beta_{1,2}^{(2)} = -\beta_{0,1}^{(2)}$,

$$s = \frac{\left[1-\beta_{0,1}^{(1,0)}\right]\left[1+\beta_{0,1}^{(2,1)}\right]}{\left[1+\beta_{0,1}^{(1,0)}\right]-\left[\beta_{0,1}^{(2)}-\beta_{0,1}^{(2,1)}\right]} \tag{3.3-122}$$

is the dimensionless, time-compression / time-stretch factor,

$$\beta_{0,1}^{(1,0)} = \frac{\hat{n}_{0,1}\bullet\mathbf{V}_{1,0}}{c} \tag{3.3-123}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_{1,0}$ in the direction $\hat{n}_{0,1}$,

$$\beta_{0,1}^{(2,1)} = \frac{\hat{n}_{0,1}\bullet\mathbf{V}_{2,1}}{c} \tag{3.3-124}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_{2,1}$ in the direction $\hat{n}_{0,1}$,

$$\beta_{0,1}^{(2)} = \frac{\hat{n}_{0,1}\bullet\mathbf{V}_2}{c} = \frac{\hat{n}_{0,1}\bullet\mathbf{V}_0}{c} \tag{3.3-125}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_2 = \mathbf{V}_0$ in the direction $\hat{n}_{0,1}$,

$$\mathbf{V}_{1,0} = \mathbf{V}_1 - (\hat{n}_{V_1}\bullet\mathbf{V}_0)\hat{n}_{V_1} = \left[V_1 - V_0(\hat{n}_{V_1}\bullet\hat{n}_{V_0})\right]\hat{n}_{V_1} \tag{3.3-126}$$

is the velocity vector of the discrete point scatterer *relative* to the velocity vector of the transmitter *in the direction of the velocity vector of the discrete point scatterer* $\hat{n}_{V_1}$,

$$\begin{aligned}\mathbf{V}_{2,1} &= \mathbf{V}_2 - (\hat{n}_{V_2} \bullet \mathbf{V}_1)\hat{n}_{V_2} = \left[V_2 - V_1(\hat{n}_{V_2} \bullet \hat{n}_{V_1})\right]\hat{n}_{V_2} \\ &= \mathbf{V}_0 - (\hat{n}_{V_0} \bullet \mathbf{V}_1)\hat{n}_{V_0} = \left[V_0 - V_1(\hat{n}_{V_0} \bullet \hat{n}_{V_1})\right]\hat{n}_{V_0}\end{aligned} \tag{3.3-127}$$

is the velocity vector of the receiver *relative* to the velocity vector of the discrete point scatterer *in the direction of the velocity vector of the receiver* $\hat{n}_{V_2} = \hat{n}_{V_0}$,

$$\tau = \frac{1}{s}\left[\tau_0 + \frac{s - 1 - \beta_{0,1}^{(2,1)}}{1 + \beta_{0,1}^{(2,1)}}\frac{r_{0,1}}{c}\right] \tag{3.3-128}$$

is the time delay in seconds,

$$\tau_0 = \frac{2\,r_{0,1}}{c} \tag{3.3-129}$$

is the time delay in seconds when *neither* the discrete point scatterer *nor* the transmitter and receiver are in motion, and $r_{0,1}$ is given by (3.3-97).

If both the transmitter and receiver are *not* in motion, then

$$\mathbf{V}_2 = \mathbf{V}_0 = \mathbf{0}. \tag{3.3-130}$$

Therefore,

$$\mathbf{V}_{1,0} = \mathbf{V}_1, \tag{3.3-131}$$

and since the relative velocity vector $\mathbf{V}_{2,1}$ given by (3.3-127) is *undefined* when $\mathbf{V}_2 = \mathbf{V}_0 = \mathbf{0}$, we set

$$\mathbf{V}_{2,1} = \mathbf{0}. \tag{3.3-132}$$

In addition (see Fig. 3.3-3),

$$r''_{1,2} = r'_{1,2} = r'_{0,1} \tag{3.3-133}$$

and

$$\hat{n}''_{1,2} = \hat{n}'_{1,2} = -\hat{n}'_{0,1}, \tag{3.3-134}$$

and from (3.3-99),

$$\mathbf{r}''_2 = \mathbf{r}_2. \tag{3.3-135}$$

With the use of (3.3-130) through (3.3-132), (3.3-134), and (3.3-135), the time-variant, space-variant, complex frequency response of the ocean given by (3.3-121) reduces to

$$H_M(t,\mathbf{r}_2\,|\,f,\mathbf{r}_0) \approx g_1(f,\hat{n}'_{0,1},-\hat{n}'_{0,1})\frac{\exp[-2\alpha(f)r_{0,1}]}{(4\pi r_{0,1})^2}\exp[+j2\pi fs(t-\tau)]\exp(-j2\pi ft), \tag{3.3-136}$$

where $\hat{n}'_{0,1}$ is given by (3.3-100), (3.3-102), and (3.3-111), $\beta_{0,1}^{(1,0)} = \beta_{0,1}^{(1)}$, $\beta_{1,2}^{(2,1)} = -\beta_{0,1}^{(2,1)} = 0$, $\beta_{1,2}^{(1,0)} = -\beta_{0,1}^{(1,0)} = -\beta_{0,1}^{(1)}$, $\beta_{1,2}^{(2)} = -\beta_{0,1}^{(2)} = 0$,

$$s = \frac{1 - \beta_{0,1}^{(1)}}{1 + \beta_{0,1}^{(1)}} \tag{3.3-137}$$

is the dimensionless, time-compression / time-stretch factor,

$$\beta_{0,1}^{(1)} = \frac{\hat{n}_{0,1} \bullet \mathbf{V}_1}{c} \tag{3.3-138}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_1$ in the direction $\hat{n}_{0,1}$,

$$\tau = \frac{1}{s}\left[\tau_0 + (s-1)\frac{r_{0,1}}{c}\right] \tag{3.3-139}$$

is the time delay in seconds,

$$\tau_0 = \frac{2\,r_{0,1}}{c} \tag{3.3-140}$$

is the time delay in seconds when *neither* the discrete point scatterer *nor* the transmitter and receiver are in motion, and $r_{0,1}$ is given by (3.3-97). Note that (3.3-136) through (3.3-140) are identical with (3.2-105) through (3.2-109), respectively, derived in Example 3.2-1.

Finally, if in addition to both the transmitter and receiver *not* being in motion, the discrete point scatterer is also *not* in motion, then

$$\mathbf{V}_1 = \mathbf{0}. \tag{3.3-141}$$

Therefore,

$$s = 1, \tag{3.3-142}$$

and from Fig. 3.3-3,

$$r'_{0,1} = r_{0,1} \tag{3.3-143}$$

and

$$\hat{n}'_{0,1} = \hat{n}_{0,1}. \tag{3.3-144}$$

With the use of (3.3-142) and (3.3-144), the *time-variant*, space-variant, complex frequency response of the ocean given by (3.3-136) reduces to the following exact *time-invariant*, space-variant, complex frequency response of the ocean:

$$H_M(t,\mathbf{r}_2\,|f,\mathbf{r}_0) = H_M(f,\mathbf{r}_2\,|\mathbf{r}_0) = g_1(f,\hat{n}_{0,1},-\hat{n}_{0,1})\frac{\exp[-2\alpha(f)r_{0,1}]}{(4\pi r_{0,1})^2}\exp(-j2\pi f\tau), \tag{3.3-145}$$

where $\hat{n}_{0,1}$ is given by (3.1-18),

$$\tau = \frac{2r_{0,1}}{c} \tag{3.3-146}$$

is the time delay in seconds, and $r_{0,1}$ is given by (3.3-97). Note that (3.3-145) and (3.3-146) are identical with (3.1-32) and (3.1-33), respectively, derived in Example 3.1-1, and with (3.2-118) and (3.2-119), respectively, derived in Example 3.2-2.

Example 3.3-2 Bistatic Scattering Geometry

We begin this example by considering the case when only the discrete point scatterer is in motion with constant velocity vector $\mathbf{V}_1$ and the scattering geometry is bistatic. Therefore,

$$\mathbf{V}_0 = \mathbf{0} \tag{3.3-147}$$

and

$$\mathbf{V}_2 = \mathbf{0}, \tag{3.3-148}$$

and as a result,

$$\mathbf{V}_{1,0} = \mathbf{V}_1, \tag{3.3-149}$$

and since the relative velocity vector $\mathbf{V}_{2,1}$ given by (3.3-94) is *undefined* when $\mathbf{V}_2 = \mathbf{0}$, we set

$$\mathbf{V}_{2,1} = \mathbf{0}. \tag{3.3-150}$$

In addition (see Fig. 3.3-3),

$$r''_{1,2} = r'_{1,2} \tag{3.3-151}$$

and

$$\hat{n}''_{1,2} = \hat{n}'_{1,2}, \tag{3.3-152}$$

and from (3.3-99),

$$\mathbf{r}''_2 = \mathbf{r}_2. \tag{3.3-153}$$

With the use of (3.3-148) through (3.3-150), (3.3-152), and (3.3-153), the time-variant, space-variant, complex frequency response of the ocean given by (3.3-87) reduces to

$$H_M(t,\mathbf{r}_2 \,|\, f,\mathbf{r}_0) \approx g_1\left(f,\hat{n}'_{0,1},\hat{n}'_{1,2}\right)\frac{\exp[-\alpha(f)(r_{0,1}+r_{1,2})]}{16\pi^2 r_{0,1} r_{1,2}}\exp[+j2\pi f s(t-\tau)]\exp(-j2\pi f t),$$

(3.3-154)

where $\hat{n}'_{0,1}$ is given by (3.2-85), (3.2-87), and (3.2-94), $\hat{n}'_{1,2}$ is given by (3.2-86), (3.2-88), and (3.2-94),

$$s = \frac{1-\beta_{0,1}^{(1)}}{1-\beta_{1,2}^{(1)}} \tag{3.3-155}$$

is the dimensionless, time-compression / time-stretch factor,

$$\beta_{0,1}^{(1)} = \frac{\hat{n}_{0,1} \bullet \mathbf{V}_1}{c} \tag{3.3-156}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_1$ in the direction $\hat{n}_{0,1}$,

$$\beta_{1,2}^{(1)} = \frac{\hat{n}_{1,2} \bullet \mathbf{V}_1}{c} \tag{3.3-157}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_1$ in the direction $\hat{n}_{1,2}$,

$$\tau = \frac{1}{s}\left[\tau_0 + (s-1)\frac{r_{1,2}}{c}\right] \tag{3.3-158}$$

is the time delay in seconds,

$$\tau_0 = \frac{r_{0,1}+r_{1,2}}{c} \tag{3.3-159}$$

is the time delay in seconds when *neither* the discrete point scatterer *nor* the transmitter and receiver are in motion, and $r_{0,1}$ and $r_{1,2}$ are given by (3.3-97) and (3.3-98), respectively. Note that (3.3-154) through (3.3-159) are identical with (3.2-77) through (3.2-82), respectively.

Finally, if the discrete point scatterer is *not* in motion, then

$$\mathbf{V}_1 = \mathbf{0}. \tag{3.3-160}$$

Therefore,

$$s = 1, \tag{3.3-161}$$

and from Fig. 3.3-3,

$$r'_{0,1} = r_{0,1}, \tag{3.3-162}$$

$$\hat{n}'_{0,1} = \hat{n}_{0,1}, \tag{3.3-163}$$

$$r'_{1,2} = r_{1,2}, \tag{3.3-164}$$

and

$$\hat{n}'_{1,2} = \hat{n}_{1,2}. \tag{3.3-165}$$

With the use of (3.3-161), (3.3-163), and (3.3-165), the *time-variant*, space-variant, complex frequency response of the ocean given by (3.3-154) reduces to the following exact *time-invariant*, space-variant, complex frequency response of the ocean:

$$H_M\left(t,\mathbf{r}_2 \,|\, f,\mathbf{r}_0\right) = H_M\left(f,\mathbf{r}_2 \,|\, \mathbf{r}_0\right) = g_1\left(f,\hat{n}_{0,1},\hat{n}_{1,2}\right)\frac{\exp\left[-\alpha(f)(r_{0,1}+r_{1,2})\right]}{16\pi^2 r_{0,1} r_{1,2}}\exp(-j2\pi f\tau), \tag{3.3-166}$$

where $\hat{n}_{0,1}$ and $\hat{n}_{1,2}$ are given by (3.1-18) and (3.1-19), respectively,

$$\tau = \frac{r_{0,1}+r_{1,2}}{c} \tag{3.3-167}$$

is the time delay in seconds, and $r_{0,1}$ and $r_{1,2}$ are given by (3.3-97) and (3.3-98), respectively. Note that (3.3-166) and (3.3-167) are identical with (3.1-17) and (3.1-20), respectively, and with (3.2-116) and (3.2-117), respectively, derived in Example 3.2-2.

Example 3.3-3 Pulse Propagation

In this example we will demonstrate how to use the pulse-propagation coupling equations given by (2.2-44) through (2.2-47) for the bistatic scattering problem shown in Fig. 3.3-1. We begin by evaluating (2.2-47), which is repeated below for convenience:

$$\tilde{y}_M(t,\mathbf{r}''_{R_n}) = \sum_{i=1}^{N_T} \int_{-\infty}^{\infty} \tilde{X}(f) c_{T_i}(f+f_c) S_{T_i}(f+f_c) H_M\left(t,\mathbf{r}''_{R_n} \middle| f+f_c,\mathbf{r}_{T_i}\right)\exp(+j2\pi f t)\,df, \tag{2.2-47}$$

where $\tilde{y}_M(t,\mathbf{r}''_{R_n})$ is the complex envelope of the acoustic signal (velocity potential) incident upon element n in the receive array at time t and position $\mathbf{r}''_{R_n} = (x''_{R_n}, y''_{R_n}, z''_{R_n})$ with units of squared-meters per second.

The time-variant, space-variant, complex frequency response of the ocean that describes the bistatic scattering problem shown in Fig. 3.3-1 is given by (3.3-87). Therefore

$$H_M\left(t,\mathbf{r}''_{R_n} \middle| f+f_c,\mathbf{r}_{T_i}\right) \approx g_1\left(f+f_c,\hat{n}'_{T_i,1},\hat{n}''_{1,R_n}\right)\frac{\exp\left[-\alpha(f+f_c)(r_{T_i,1}+r_{1,R_n})\right]}{16\pi^2 r_{T_i,1} r_{1,R_n}}\exp\left[+j2\pi(f+f_c)s_{T_i,R_n}(t-\tau_{T_i,R_n})\right]$$
$$\times\exp\left[-j2\pi(f+f_c)t\right],$$

(3.3-168)

where

$$s_{T_i,R_n} = \frac{\left[1-\beta_{T_i,1}^{(1,0)}\right]\left[1-\beta_{1,R_n}^{(2,1)}\right]}{\left[1-\beta_{1,R_n}^{(1,0)}\right]+\left[\beta_{1,R_n}^{(2)}-\beta_{1,R_n}^{(2,1)}\right]} \tag{3.3-169}$$

is the dimensionless, time-compression / time-stretch factor [see (3.3-88)],

$$\beta_{T_i,1}^{(1,0)} = \frac{\hat{n}_{T_i,1} \bullet \mathbf{V}_{1,0}}{c} \tag{3.3-170}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_{1,0}$ in the direction $\hat{n}_{T_i,1}$ [see (3.3-89) and (3.1-37)],

$$\beta_{1,R_n}^{(2,1)} = \frac{\hat{n}_{1,R_n} \bullet \mathbf{V}_{2,1}}{c} \tag{3.3-171}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_{2,1}$ in the direction $\hat{n}_{1,R_n}$ [see (3.3-90) and (3.1-38)],

$$\beta_{1,R_n}^{(1,0)} = \frac{\hat{n}_{1,R_n} \bullet \mathbf{V}_{1,0}}{c} \tag{3.3-172}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_{1,0}$ in the direction $\hat{n}_{1,R_n}$ [see (3.3-91) and (3.1-38)],

$$\beta_{1,R_n}^{(2)} = \frac{\hat{n}_{1,R_n} \bullet \mathbf{V}_2}{c} \tag{3.3-173}$$

is the dimensionless, normalized radial-component of $\mathbf{V}_2$ in the direction $\hat{n}_{1,R_n}$ [see (3.3-92) and (3.1-38)],

$$\mathbf{V}_{1,0} = \mathbf{V}_1 - (\hat{n}_{V_1} \bullet \mathbf{V}_0)\hat{n}_{V_1} = \left[V_1 - V_0(\hat{n}_{V_1} \bullet \hat{n}_{V_0})\right]\hat{n}_{V_1} \tag{3.3-174}$$

is the velocity vector of the discrete point scatterer *relative* to the velocity vector of the transmitter *in the direction of the velocity vector of the discrete point scatterer* $\hat{n}_{V_1}$ [see (3.3-93)],

$$\mathbf{V}_{2,1} = \mathbf{V}_2 - (\hat{n}_{V_2} \bullet \mathbf{V}_1)\hat{n}_{V_2} = \left[V_2 - V_1(\hat{n}_{V_2} \bullet \hat{n}_{V_1})\right]\hat{n}_{V_2} \tag{3.3-175}$$

is the velocity vector of the receiver *relative* to the velocity vector of the discrete point scatterer *in the direction of the velocity vector of the receiver* $\hat{n}_{V_2}$ [see (3.3-94)],

$$\tau_{T_i,R_n} = \frac{1}{s_{T_i,R_n}}\left[\tau_0 + \frac{s_{T_i,R_n} - 1 + \beta_{1,R_n}^{(2,1)}}{1-\beta_{1,R_n}^{(2,1)}}\frac{r_{1,R_n}}{c}\right] \tag{3.3-176}$$

is the time delay in seconds (the amount of time it takes for the acoustic signal transmitted from element i in the transmit array to *begin* to appear at element n in the receive array) [see (3.3-95)],

$$\tau_0 = \frac{r_{T_i,1} + r_{1,R_n}}{c} \tag{3.3-177}$$

is the time delay in seconds when *neither* the discrete point scatterer *nor* the transmit and receive arrays are in motion [see (3.3-96)], and

$$r_{T_i,1} = \left|\mathbf{r}_{T_i,1}\right| = \left|\mathbf{r}_1 - \mathbf{r}_{T_i}\right| \tag{3.3-178}$$

and

$$r_{1,R_n} = \left|\mathbf{r}_{1,R_n}\right| = \left|\mathbf{r}_{R_n} - \mathbf{r}_1\right| \tag{3.3-179}$$

are the distances (ranges) in meters measured from element i in the transmit array to the discrete point scatterer, and from the discrete point scatterer to element n in the receive array, respectively [see (3.3-97) and (3.3-98)], when motion begins at time $t = t_0 = 0$ seconds.

The position vector from the origin to element n in the receive array when the scattered acoustic field is incident upon element n is given by [see (3.3-99)]

$$\mathbf{r}''_{R_n} = \mathbf{r}_{R_n} + \Delta t'\mathbf{V}_2 + (t - \Delta t')\mathbf{V}_{2,1}, \qquad t \geq \tau_{T_i,R_n}, \tag{3.3-180}$$

where $\mathbf{r}_{R_n}$ is the position vector from the origin to element n in the receive array when motion begins at time $t = t_0 = 0$ seconds, and [see (3.3-71)]

$$\Delta t' \approx \frac{1}{1 - \beta^{(1,0)}_{1,R_n} + \beta^{(2)}_{1,R_n} - \beta^{(2,1)}_{1,R_n}} \left\{\left[1 - \beta^{(2,1)}_{1,R_n}\right]t - \frac{r_{1,R_n}}{c}\right\}, \qquad t \geq \tau_{T_i,R_n}. \tag{3.3-181}$$

In addition, in order to evaluate the scattering amplitude function, we need the following equations:

$$\hat{n}'_{T_i,1} = \frac{\mathbf{r}'_{T_i,1}}{\left|\mathbf{r}'_{T_i,1}\right|} \tag{3.3-182}$$

is the dimensionless unit vector in the direction measured from element i in the transmit array to the discrete point scatterer when the transmitted acoustic field is first incident upon the discrete point scatterer at time t' [see (3.3-100)], and

$$\hat{n}''_{1,R_n} = \frac{\mathbf{r}''_{1,R_n}}{\left|\mathbf{r}''_{1,R_n}\right|} \tag{3.3-183}$$

is the dimensionless unit vector in the direction measured from the discrete point scatterer to element n in the receive array when the scattered acoustic field is first incident upon element n [see (3.3-101)], where [see (3.3-102)]

$$\mathbf{r}'_{T_i,1} = \mathbf{r}_{T_i,1} + \Delta t' \mathbf{V}_{1,0} \tag{3.3-184}$$

and [see (3.3-103)]

$$\mathbf{r}''_{1,R_n} = \mathbf{r}_{1,R_n} + \Delta t'(\mathbf{V}_2 - \mathbf{V}_{1,0}) + \Delta t'' \mathbf{V}_{2,1}, \tag{3.3-185}$$

where [see (3.1-4)]

$$\mathbf{r}_{T_i,1} = \mathbf{r}_1 - \mathbf{r}_{T_i} \tag{3.3-186}$$

and [see (3.1-10)]

$$\mathbf{r}_{1,R_n} = \mathbf{r}_{R_n} - \mathbf{r}_1 \tag{3.3-187}$$

are the position vectors measured from element i in the transmit array to the discrete point scatterer, and from the discrete point scatterer to element n in the receive array, respectively, when motion begins at time $t = t_0 = 0$ seconds, where [see (3.3-111)]

$$\Delta t' \approx \frac{1}{1 - \beta^{(1,0)}_{1,R_n} + \beta^{(2)}_{1,R_n} - \beta^{(2,1)}_{1,R_n}} \left\{ \left[1 - \beta^{(2,1)}_{1,R_n}\right] \tau_{T_i,R_n} - \frac{r_{1,R_n}}{c} \right\} \tag{3.3-188}$$

and [see (3.3-108)]

$$\Delta t'' = \tau_{T_i,R_n} - \Delta t', \tag{3.3-189}$$

where $\Delta t'$ is given by (3.3-188). Note that the parameter $\Delta t'$ given by (3.3-181) is a function of time t whereas $\Delta t'$ given by (3.3-188) depends on the time-delay τ_{T_i,R_n}.

Substituting (3.3-168) into (2.2-47) yields

$$\tilde{y}_M(t,\mathbf{r}''_{R_n}) \approx \sum_{i=1}^{N_T} \int_{-\infty}^{\infty} \tilde{X}'_{T_i,R_n}(f) \exp\left[+j2\pi f s_{T_i,R_n}(t - \tau_{T_i,R_n})\right] df \exp\left[+j2\pi f_c s_{T_i,R_n}(t - \tau_{T_i,R_n})\right] \exp(-j2\pi f_c t), \tag{3.3-190}$$

or

$$\tilde{y}_M(t,\mathbf{r}''_{R_n}) \approx \sum_{i=1}^{N_T} \tilde{x}'_{T_i,R_n}(s_{T_i,R_n}[t - \tau_{T_i,R_n}]) \exp\left[+j2\pi f_c (s_{T_i,R_n} - 1)t\right] \exp(-j2\pi f_c s_{T_i,R_n} \tau_{T_i,R_n}), \tag{3.3-191}$$

where

$$\tilde{X}'_{T_i,R_n}(f) = \tilde{X}(f) c_{T_i}(f + f_c) S_{T_i}(f + f_c) g_1\left(f + f_c, \hat{n}'_{T_i,1}, \hat{n}''_{1,R_n}\right) \frac{\exp\left[-\alpha(f + f_c)(r_{T_i,1} + r_{1,R_n})\right]}{16\pi^2 r_{T_i,1} r_{1,R_n}} \tag{3.3-192}$$

and

$$\tilde{x}'_{T_i,R_n}(t) = F_f^{-1}\left\{\tilde{X}'_{T_i,R_n}(f)\right\} = \int_{-\infty}^{\infty} \tilde{X}'_{T_i,R_n}(f)\exp(+j2\pi ft)\,df. \tag{3.3-193}$$

Equation (3.3-191) indicates that the complex envelope, $\tilde{y}_M(t,\mathbf{r}''_{R_n})$, of the acoustic signal (velocity potential) incident upon element n in the receive array is equal to the sum of time-compressed / time-stretched, time-delayed, Doppler shifted, modified versions of the complex envelopes of the transmitted electrical signals from each element in the transmit array. The amount of Doppler shift in hertz is given by

$$\phi_{T_i,R_n} = (s_{T_i,R_n} - 1)f_c. \tag{3.3-194}$$

Equation (3.3-192) shows how the frequency spectrum of the complex envelope of the transmitted electrical signal, $\tilde{X}(f)$, gets modified by various other important frequency-dependent functions in order to produce the complex envelope, $\tilde{y}_M(t,\mathbf{r}''_{R_n})$, of the acoustic signal (velocity potential) incident upon element n in the receive array.

Before continuing with the analysis, several comments concerning the dimensionless, time-compression / time-stretch factor s_{T_i,R_n} and (3.3-194) are in order. There are *three* ranges of values for s_{T_i,R_n}; namely, $s_{T_i,R_n}=1$, $s_{T_i,R_n}>1$, and $s_{T_i,R_n}<1$. When $s_{T_i,R_n}=1$, this is an indication that there is *no* time compression or time stretch, and as a result, there is *no* change in signal bandwidth. Also, when $s_{T_i,R_n}=1$, $\phi_{T_i,R_n}=0$; that is, there is *no* Doppler shift [see (3.3-194)]. For example, if there is *no* motion, then all three velocity vectors are equal to the zero vector. Therefore, all four beta parameters are equal to zero and, as a result, $s_{T_i,R_n}=1$. When $s_{T_i,R_n}>1$, this is an indication of *time compression*, that is, the duration of the received signal given by (3.3-191) is *decreased* (compared to the duration of the transmitted signal) resulting in an *increase* in signal bandwidth. Also, when $s_{T_i,R_n}>1$, $\phi_{T_i,R_n}>0$; which indicates a *positive* Doppler shift [see (3.3-194)]. Finally, when $s_{T_i,R_n}<1$, this is an indication of *time stretch*, that is, the duration of the received signal is *increased* (compared to the duration of the transmitted signal) resulting in a *decrease* in signal bandwidth. Also, when $s_{T_i,R_n}<1$, $\phi_{T_i,R_n}<0$; which indicates a *negative* Doppler shift [see (3.3-194)].

If we next substitute (3.3-191) into (2.2-46), then we obtain

$$y_M(t,\mathbf{r}''_{R_n}) \approx \sum_{i=1}^{N_T} \operatorname{Re}\left\{\tilde{x}'_{T_i,R_n}(s_{T_i,R_n}[t-\tau_{T_i,R_n}])\exp\left[+j2\pi f_c s_{T_i,R_n}(t-\tau_{T_i,R_n})\right]\right\}. \tag{3.3-195}$$

Since $x(t)$ given by (2.2-28) and $\tilde{x}(t)$ given by (2.2-30) are related by

$$x(t) = \operatorname{Re}\{\tilde{x}(t)\exp(+j2\pi f_c t)\}, \tag{3.3-196}$$

if we let

$$x'_{T_i,R_n}(t) = \operatorname{Re}\left\{\tilde{x}'_{T_i,R_n}(t)\exp(+j2\pi f_c t)\right\}, \tag{3.3-197}$$

then (3.3-195) can be rewritten as

$$y_M(t, \mathbf{r}''_{R_n}) \approx \sum_{i=1}^{N_T} x'_{T_i,R_n}(s_{T_i,R_n}[t - \tau_{T_i,R_n}]). \tag{3.3-198}$$

Substituting (3.3-198) into (2.2-45) yields

$$Y_M(\eta, \mathbf{r}''_{R_n}) \approx \sum_{i=1}^{N_T} \frac{1}{|s_{T_i,R_n}|} X'_{T_i,R_n}(\eta/s_{T_i,R_n}) \exp(-j2\pi\eta\tau_{T_i,R_n}), \tag{3.3-199}$$

and upon substituting (3.3-199) into (2.2-44), we obtain

$$y(t, \mathbf{r}''_{R_n}) \approx \sum_{i=1}^{N_T} \int_{-\infty}^{\infty} X''_{T_i,R_n}(\eta) \exp\left[+j2\pi\eta(t - \tau_{T_i,R_n})\right] d\eta, \qquad n = 1,2,\ldots,N_R, \tag{3.3-200}$$

or, finally,

$$y(t, \mathbf{r}''_{R_n}) \approx \sum_{i=1}^{N_T} x''_{T_i,R_n}(t - \tau_{T_i,R_n}), \qquad n = 1,2,\ldots,N_R, \tag{3.3-201}$$

where

$$X''_{T_i,R_n}(\eta) = \frac{1}{|s_{T_i,R_n}|} X'_{T_i,R_n}(\eta/s_{T_i,R_n}) c_{R_n}(\eta) S_{R_n}(\eta) \tag{3.3-202}$$

and

$$x''_{T_i,R_n}(t) = F_\eta^{-1}\left\{X''_{T_i,R_n}(\eta)\right\} = \int_{-\infty}^{\infty} X''_{T_i,R_n}(\eta) \exp(+j2\pi\eta t) d\eta. \tag{3.3-203}$$

Equation (3.3-201) indicates that the output electrical signal $y(t, \mathbf{r}''_{R_n})$ from element n in the receive array at time t and position $\mathbf{r}''_{R_n} = (x''_{R_n}, y''_{R_n}, z''_{R_n})$ with units of volts is equal to the sum of time-delayed, modified versions of the transmitted electrical signals from each element in the transmit array.

4 Summary

A set of pulse-propagation coupling equations was successfully derived. They couple the output electrical signal at a point element in a receive array to the transmitted electrical signal at the input to a transmit array via the complex frequency response of a fluid medium (e.g., air or water). The pulse-propagation coupling equations are based on linear, time-variant, space-variant, filter theory, the principles of complex aperture theory and array theory, and solving a linear wave equation, which includes satisfying all boundary conditions, including the boundary condition at the source. The time-variant, space-variant, complex frequency response of the ocean was shown to be the time-harmonic solution of a linear wave equation when the source distribution is a time-harmonic point source.

The pulse-propagation coupling equations provide a consistent, logical, and straightforward mathematical framework that can be used to accurately model the propagation of small-amplitude acoustic pulses in the ocean for a bistatic scattering problem. The main features of the pulse-propagation coupling equations are as follows: 1) transmitted electrical signals are modeled as amplitude-and-angle-modulated carriers, 2) both the transmit and receive apertures are modeled as volume, conformal arrays composed of unevenly-spaced, complex-weighted, point elements (this type of model for both of the apertures allows for maximum flexibility), 3) the complex weights are frequency dependent and allow for beamforming, 4) the performance of the point elements in both the transmit and receive arrays are characterized by frequency-dependent, transmitter and receiver sensitivity functions, and 5) the solution of a linear wave equation is given by the complex frequency response of the fluid medium. It is important to note that attention to all proper units of measurement were taken into account in order to ensure the accurate prediction of signal strength levels at each element in a receive array. This is especially important, for example, in order to obtain accurate probability of detection results.

Derivations of the complex frequency response of the ocean for the following three different bistatic scattering problems were performed: 1) no motion, 2) only the discrete point scatterer is in motion, and 3) all three platforms (the transmitter, discrete point scatterer, and receiver) are in motion. Specific examples on the use of the pulse-propagation coupling equations were given for the three different bistatic scattering problems. Scatter from a discrete point scatterer was modeled via the scattering amplitude function, which is a complex function (magnitude and phase) and is, in general, a function of frequency, the direction of wave propagation from the source to the scatterer, and the direction of wave propagation from the scatterer to the receiver. In addition to the scattering amplitude function, frequency-dependent attenuation was taken into account in order to model the propagation of sound from transmitter to discrete point scatterer, and from discrete point scatterer to receiver. The dimensionless, time-compression / time-stretch factor was derived and discussed for the two bistatic scattering problems involving motion. The time-compression / time-stretch factor takes into account the relativistic effects of motion and provides for accurate time delay and Doppler shift values.

References

[1] L. J. Ziomek, *Fundamentals of Acoustic Field Theory and Space-Time Signal Processing*, CRC Press, Inc., Boca Raton, Florida, 1995, pp. 651-654.

[2] P. M. Morse and K. U. Ingard, *Theoretical Acoustics*, McGraw-Hill, New York, 1968, pg. 322.

[3] see [1], pg. 131.

[4] see [2], pg. 319.

[5] see [1], pp. 124-126.

[6] see [1], pp. 204-221.

[7] see [1], pp. 222-240.

[8] see [1], pp. 240-256

[9] see [1], pp. 401-405

[10] see [1], pp. 605-608.

[11] see [1], pp. 612-613.

[12] A. Ishimaru, *Wave Propagation and Scattering in Random Media*, Volume 1, Academic Press, New York, 1978, pp. 9-12, 39-40.

[13] C. S. Clay and H. Medwin, *Acoustical Oceanography: Principles And Applications*, Wiley, New York, 1977, pp. 180-184.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Technical Information Center 8725 John J. Kingman Rd., STE 0944 Ft. Belvoir, VA 22060-6218	2
2.	Dudley Knox Library, Code 52 Naval Postgraduate School 411 Dyer Road Monterey, CA 93943-5101	2
3.	Research Office, Code 09 Naval Postgraduate School 589 Dyer Road Monterey, CA 93943-5138	1
4.	Chairman, Code EC Department of Electrical and Computer Engineering Naval Postgraduate School 833 Dyer Road Monterey, CA 93943-5121	1
5.	Professor Lawrence J. Ziomek, Code EC/Zm Department of Electrical and Computer Engineering Naval Postgraduate School 833 Dyer Road Monterey, CA 93943-5121	3
6.	Darrell B. Spires Program Executive Office for Mine and Undersea Warfare Attn: PMS403D2 614 Sicard Street SE STOP 7014 Washington Navy Yard DC 20376-7014	3
7.	Michael R. Medeiros Naval Undersea Warfare Center 1176 Howell Street Code 8221, Bldg 149 Newport, RI 02841	3